1. Let $|G|=20$. Then $n_5 \equiv 1 \mod 5$ and $n_5 | 4$. Thus $n_5=1$, so the Sylow 5 subgroup is normal, $Z_5 \trianglelefteq G$. Let $P \in Syl_2(G)$ so $|P|=4$. Then $P \cap Z_5 = \{e\}$ and $G = PZ_5$ so P is a complement to Z_5, thus $G \cong Z_5 \rtimes P$.

Let $Z_5 = \langle g \mid g^5 = e \rangle$. Recall $Aut(Z_5) \cong Z_4 = \langle \varphi \rangle$ where $\varphi(g) = g^2$.

Case 1 $P \cong Z_4 = \langle x \mid x^4 = e \rangle$. Then there are 3 maps $\varphi: P \to Aut Z_4$, namely $\varphi_1: x \to e$, $\varphi_2: x \to \varphi^2$, $\varphi_3: x \to \varphi$, so φ_3 is an $\cong$, $\ker \varphi_2 = \{e, x^2\}$ and φ_1 is trivial. Thus

$$Z_5 \rtimes_{\varphi_1} Z_4 \cong Z_5 \times Z_4 = \langle x, g \mid x^4 = g^5 = e,\ xg = gx \rangle$$

$$Z_5 \rtimes_{\varphi_2} Z_4 = \langle x, g \mid x^4 = g^5 = e,\ xgx^{-1} = g^4 \rangle$$

$$Z_5 \rtimes_{\varphi_3} Z_4 = \langle x, g \mid x^4 = g^5 = e,\ xgx^{-1} = g^2 \rangle$$

Case 2 $P \cong Z_2 \times Z_2$. Since P has no elements of order 4, any $\varphi: P \to Aut Z_5$ must have image $\{e\}$ or $\{e, \varphi^2\}$. If $\varphi = id$ then

$$G = Z_2 \times Z_2 \times Z_5 = \langle x, y, g \mid x^2 = y^2 = g^5 = e,\ xg = gx,\ xy = yx,\ yg = gy \rangle$$

Else let $\ker \varphi = \langle x \rangle$ and $y \notin \ker \varphi$. So $P = \langle x, y \mid x^2 = y^2 = e, xy = yx \rangle$ and $\varphi(x) = id$, $\varphi(y) = \varphi^2$

$$G = \langle x, y, g \mid x^2 = y^2 = g^5 = e,\ xg = gx,\ ygy^{-1} = g^4 \rangle$$

$\boxed{\text{5 groups up to } \cong}$

2a. $|G| < \infty$. TFAE

1. G is nilpotent
2. $G \cong$ to $\times$ of it's Sylows
3. all Sylows are normal
4. elements of coprime order commute
5. Proper subgroups are proper in their normalizers

b. Yes! $1 \lhd V \lhd A_4 \lhd S_4$, each quotient is abelian

3. Let ~~$x = (12)(34)$, $y = (123)$. Then~~ $xy = (243)$.

~~$A_4 = \langle x, y \mid x^2 = y^3 = (xy)^3 = e \rangle$~~

Let $x = (123)$ $y = (234)$

$xy = (12)(34)$
$yx = (13)(24)$
$xyyx = (14)(23)$

$$\boxed{A_4 = \langle x, y \mid x^3 = y^3 = (xy)^2 = (yx)^2 = (xyyx)^2 = e \rangle}$$

4. Suppose $H, K \trianglelefteq G$, H nilp of class h, K of class k. Then HK is nilpotent of class $\leq h+k$

Proof Recall $G^1 = [G,G]$, $G^2 = [G,G^1]$ $G^i = [G,G^{i-1}]$ ...

Lemma

$$G^n \subseteq H^n K^n (H^0 \cap K^{n-1})(H^1 \cap K^{n-2}) \cdots (H^{n-1} \cap K^0)$$

Rmk Lemma $\Rightarrow G^{h+k} = \{e\}$

Proof Note that $[HK, L] = [H,L][K,L]$ so

$$[G, H^{n-1}] = [H, H^{n-1}][K, H^{n-1}]$$
$$\subseteq H^n (H^{n-1} \cap K)$$
$$[G, K^{n-1}] \subseteq K^n (H \cap K^{n-1})$$

$$[G, H^i \cap K^j] = [H, H^i \cap K^j][K, H^i \cap K^j]$$
$$\subseteq (H^{i} \cap K^j)(H^i \cap K^{j+1})$$

now induce

5. The elements of $F(S)$ are finite reduced words in $\{s \mid s\in S\} \cup \{s^{-1} \mid s\in S\} \cup \{1\}$. The operation is concatenation followed by reducing (cancelling $s s^{-1}$ and eliminating extraneous 1's.

<u>Universal Property</u>. Let $\varphi: S \to G$ be any map of sets, where G is a group. Then there exists a unique group homomorphism $\tilde{\varphi}: F(S) \to G$ such that $\tilde{\varphi}(s) = \varphi(s)$ $\forall s \in S$.

6. Let $K, H, N \trianglelefteq G$ be nontrivial, so $G \cong H \times K$. Prove N is in the center or that N intersects one of H & K nontrivially

<u>Proof</u> Suppose $N \cap H = \{e\}$ and $N \cap K = \{e\}$. We must prove N is in the center. Suppose not, let $n = (h,k) \notin Z(G)$. So $\exists (h_1,k_1) \in G$ so that $n(h_1,k_1) \neq (h_1,k_1)n$. Then

$$(h,k)(h_1,k_1)(h^{-1},k^{-1})(h_1^{-1},k_1^{-1}) \neq (e,e)$$

$$(hh_1h^{-1}h_1^{-1}, kk_1k^{-1}k_1^{-1}) \neq e,e$$

So either

7. Let $H \trianglelefteq G$, $H \cap G' = \{e\}$ Prove $H \leq Z(G)$!

<u>Proof</u> Suppose $H \not\leq Z(G)$. Then $\exists h \in H, g \in G \mid hg \neq gh$. Thus $hgh^{-1}g^{-1} \neq e$. but $H \trianglelefteq G$ so $gh^{-1}g^{-1} \in H$ so

$$e \neq hgh^{-1}g^{-1} \in H \cap G'. \quad //$$

8. a. Yes! $SL_n(F) \trianglelefteq GL_n(F)$! Let $K = \left\{ \begin{pmatrix} a & & 0 \\ & 1 & \\ 0 & & \ddots & \\ & & & 1 \end{pmatrix} \middle| a \neq 0 \right\}$

Then $K \leq GL_n(F)$, $K \cap SL_n(F) = \{e\}$ Also for $A \in GL_n(F)$ we have

$$A = \underbrace{A \cdot \begin{pmatrix} \frac{1}{\det A} & & 0 \\ & 1 & \\ 0 & & \ddots & \\ & & & 1 \end{pmatrix}}_{\in SL_n(F)} \cdot \underbrace{\begin{pmatrix} \det A & & 0 \\ & 1 & \\ 0 & & \ddots & \\ & & & 1 \end{pmatrix}}_{\in K}$$

Thus $GL_n(F) = SL_n(F)K$ so

$$GL_n(F) \cong SL_n(F) \rtimes K!$$

b. Yes, let $K = \{e, (12)\}$ Then $K \cap A_n = \{e\}$ $A_n \trianglelefteq S_n$ and $S_n = KA_n$ so

$$S_n \cong A_n \rtimes Z_2$$

c. No, K would have to have 2 elements. The only subgroup of Q_8 is $\{\pm 1\}$ and this is in every 4 elt subgroup so Z_4 has no complement.