1/16/07 # Modules

Def. Let R be a ring. A left R-module is an abelian group M and an action $R \times M \to M$

$(r,m) \to rm$ such that

1. $r(m_1+m_2) = rm_1 + rm_2$
2. $(r_1+r_2)m = r_1m + r_2m$ $\quad \forall r, r_1, r_2 \in R$
3. $r_1(r_2 m) = (r_1r_2)m$ $\quad m, m_1, m_2 \in M$

*4. $1m = m$ if R has unit. (sometimes called unital)

Similarly for right modules.

Rmk If R is commutative and M is a left R-module then we can define

$mr := rm$ to make M a right module

so no difference.

If R is noncommutative then left & right modules are truly different.

Example Suppose R is a field. Then

R-modules $\longleftrightarrow$ Vector space / R.

Action is scalar multiplication

Def. A submodule is a subgroup of M closed under operation

Example: trivially 0 & M are submodules

Prop $N \subset M$ is a submodule $\leftrightarrow$ $rn_1 + n_2 \in N$ $\forall r \in R$, $n_i \in N$

Examples

1. Left regular module ${}_R R$

   - $M = R$
   - action is ring mult.
   - submodules $\longleftrightarrow$ left ideals

Similarly for right regular module R_R, may be very different.

2. Generalize #1

   Free module of rank n $\quad R^n = \{(r_1, r_2, \ldots, r_n) \mid r_i \in R\}$

   componentwise action

   $$r(r_1, r_2, \ldots, r_n) = (rr_1, \ldots, rr_n)$$

   * Generalizes $\mathbb{R}^n$ vector space *

3. Suppose R a ring w/ 1, S a subring (same identity)

   Then R is an S-module.

   Ex $\mathbb{R}$ is a $\mathbb{Z}$-module.

4. Prop The category of $\mathbb{Z}$-modules is the same as abelian groups

   $\mathbb{Z}$-module = Abelian Group

Def. Let I be a 2-sided ideal of R such that $iM=0$ $\forall i\in I$, say I annihilates M.

Def. The annihilator of M is $\{a\in R \mid am=0 \ \forall m\in M\}$

Lemma ann M is a 2-sided ideal!

Prop Suppose I annihilates M. Then M is an R/I module $(r+I)m := rm$.

Modules for the ring $F[x]$.

Fix a vector space V over F and a linear map $T: V\to V$. Define $T^2 = T\circ T$, etc.

Recall! $\mathrm{End}_F(V) = \{\text{linear maps } V\to V\}$ is a vector space.

Def. Make V into an $F[x]$ module by:

$$(a_n x^n + \dots + a_1 x + a_0)\vec{v} = a_n T^n(\vec{v}) + \dots + a_1 T(\vec{v}) + a_0\vec{v}$$

Rmk Given V, there are many $F[x]$ module structures on V depending on choice of T.

Prop Every $F[x]$ module is of this form.

Proof 1. $F \subset F[x]$ so $F[x]$ module is an F-module.

2. X acts linearly

* Submodules of V

Goal 1. Completely classify $F[x]$ modules via Jordan Canonical Form.

2. Completely understand finitely generated modules over any PID (Generalize Basis thm for $V(s)$)

– COR f.g. abelian groups

3 When $R = FG \Rightarrow$ representation theory.

Def. R a comm ring w/ 1. An R-algebra is a ring R w/ 1 together w/ a ring homo $f: R \to A$ mapping $1_R \to 1_A$ such that the subring $f(R) \subseteq$ center of A.

RMK A has a left/right R-structure $ra = ar := f(r)a$

Example $F[x]$ is an F-algebra.

$\mathbb{Z}[x]$ is $\mathbb{Z}$-algebra

Any ring w/ identity is a $\mathbb{Z}$-algebra

(Think of R as "scalars")