p. 350 #7, 8, 9, 10, 13

1/18/07

Review Ring R, abelian group $(M,+)$ is a left R-module means R acts on M, rm, so

$$(r_1 r_2)m = r_1(r_2 m)$$
$$(r_1+r_2)m = r_1 m + r_2 m \qquad r(m_1+m_2) = rm_1 + rm_2$$
$$1m = m \text{ if applicable}$$

Examples

1. F-modules are vector spaces, action is "scalar multiplication"
2. $\mathbb{Z}$-modules are abelian groups, action is "group operation"

i.e.

$$nm = \underbrace{m+m+\cdots+m}_{n \text{ times}} \qquad n>0$$
$$-nm = -m-m\cdots-m$$

3. Regular module ${}_R R$, $M = R$

4. $F[x]$ modules $\longleftrightarrow$ Vector space V together w/ linear map $T: V \to V$ such that $x \cdot \vec{v} = T(\vec{v})$.

5. Submodules of M are subgroups closed under operation of R.

R-algebras

Suppose R is commutative w/ 1. An R-algebra is a ring A w/ 1 and a ring homo $f: R \to A$ s.t.

1. $1_R \xrightarrow{f} 1_A$
2. $f(R) \subseteq Z(A)$

Think of $f(R)$ as "scalars" inside A.

Remarks / Examples

1. A becomes an R-module via $r \cdot a = f(r)a$

2. Any ring w/ 1 is a $\mathbb{Z}$ algebra.

3. $M_{n \times n}(F)$ is an F algebra, $F[x]$ is an F algebra.

4. R any subring of $Z(A)$ w/ $1_A \in R$ then A is an R-algebra.
   when R is a field then f is 1-1 so $F \subseteq A$.

Quotient Modules & Module Homos

Def Let M & N be R-modules. An R-module homomorphism is a group homomorphism $\varphi : M \to N$ such that

$$\varphi(rm) = r\,\varphi(m) \quad \forall m \in M.$$

If φ is a bijection, say it is an isomorphism and $M \cong N$.
Say M & N are isomorphic if $\exists$ an isomorphism.

Def Kernel $\varphi = \{ m \in M \mid \varphi(m) = 0 \}$
Image $\varphi(M) = \{ \varphi(m) \mid m \in M \}$

Prop These are both submodules (of M & N respectively)

Def $\mathrm{Hom}_R(M,N) = \{ R\text{-module homos } M \to N \}$

Examples / Warnings

1. May be many group homos $M \to N$ which are not R-module maps (except if $R = \mathbb{Z}$)

2. Consider ${}_R R$ left R-module. Then R-module homos $R \to R$ are different than ring homos $R \to R$

MODULE HOMOS $\quad f(r_1 + r_2) = f(r_1) + f(r_2)$

$\quad f(rs) = r f(s) \quad \forall r, s \in R$

Notice if R has an identity then

$f(r) = r f(1)$.

EX $\quad f(x) = 2x : \mathbb{Z} \to \mathbb{Z}$

$f(ab) = 2ab \quad$ Not a ring homo.

$\overset{\parallel}{a f(b)}$

3. Projection Maps $R^n \to R$

4. F-module homos $\longleftrightarrow$ linear transformations

Prop

1. $Hom_R(M,N)$ is an abelian group

2. If R is commutative then $Hom_R(M,N)$ is an R-module via

$$(rf)(m) = r(f(m))$$

3. composition : $Hom_R(M,N) \times Hom_R(L,M) \to Hom_R(L,N)$

4. $\mathrm{Hom}_R(M,M)$ aka $\mathrm{End}_R(M)$ is a ring w/ identity.
 - if R is commutative it is an R-algebra.

Proof ...

Def $\mathrm{Hom}_R(M,M)$ is endomorphism ring of M.

Example Let R be a ring w/ unity. $\mathrm{End}_R({}_R R) \cong R^{op}$

Proof ...

Thm $N \subset M$ R-modules. Then there is an R-module structure on M/N given by

$$r(m+N) = rm + N.$$ Quotient module

All isomorphisms Thms hold for free!

Def Let N_1, N_2 submodules of M.

$$N_1 + N_2 =: \{n_1 + n_2 \mid n_i \in N_i\}$$ sum of submodules is a submod.

$\cong$ Theorems

1. $\varphi: M \to N$ $\quad M/\ker\varphi \cong \mathrm{Im}\,\varphi$
2. $A, B \leq M$ then $A+B/B \cong A/A\cap B$
3. $A \leq B \leq M$. Then $M/A \big/ B/A \cong M/B$
4. Submodule lattice of M/N = lattice of subs of M containing N.