

1/23/07

Recall  $M, N$   $R$ -module. A module homomorphism is a map  $f: M \rightarrow N$  such that

- $f(m_1 + m_2) = f(m_1) + f(m_2) \quad \forall m_1, m_2 \in M$
- $f(rm) = rf(m) \quad \forall r \in R, m \in M$

i.e.

$$\begin{array}{ccc}
 m & \xrightarrow{f} & f(m) \\
 r \cdot \downarrow & & \downarrow r \cdot \\
 rm & \xrightarrow{f} & rf(m) \\
 & & f(rm)
 \end{array}$$

" $f$  commutes with actions of  $R$ "

$\text{Hom}_R(M, N)$  is set of module homomorphisms from  $M \rightarrow N$ .  
What structure does it have?

- $\text{Hom}_R(M, N)$  is an abelian group
- $R$ -commutative  $\Rightarrow \text{Hom}_R(M, N)$  is an  $R$ -module via  
 $(rf)(m) = r \cdot f(m)$
- $\text{End}_R(M) := \text{Hom}_R(M, M)$  is a ring w/  $1$ . It is a  $R$ -algebra if  $R$  is commutative.

When  $R$  is a field then  $\text{End}_F(M)$  are scalar transformations!

### Quotient Modules

Let  $N \subseteq M$  be a submodule. Then  $M/N$  is an abelian group.

Prop  $M/N$  is an  $R$ -module via  $r(m+N) := rm+N$

Proof Need to check well-defined!

In particular  $\pi: M \rightarrow M/N$  is a module homomorphism w/ kernel  $N$ .

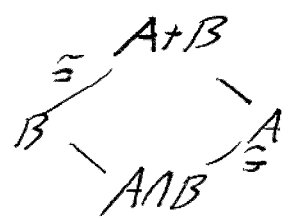
Summary Groups: Normal subgroups  $\longleftrightarrow$  kernel of a group homomorphism  
 Rings: Ideals  $\longleftrightarrow$  kernels of ring homomorphisms  
 R-modules: submodules  $\longleftrightarrow$  kernels of module homomorphisms

Def. Let  $N_1, N_2$  be submodules of  $M$ . Then  
 $N_1 + N_2 = \{n_1 + n_2 \mid n_i \in N_i\}$  is a submodule, sum of 2 submodules

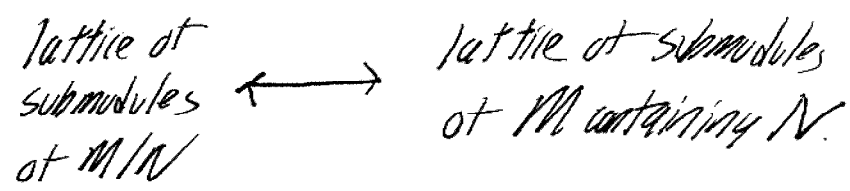
Prop. Module homomorphisms are all homs of abelian groups, so  $\cong$ 's hold. Just check the  $\cong$ 's are actually R-module maps

Isomorphism Theorems  $M, N$  R-modules

1.  $\varphi: M \rightarrow N$  then  $M/\ker \varphi \cong \varphi(M)$
2. Suppose  $A, B$  submodules of  $M$ . Then  $(A+B)/B \cong A/(A \cap B)$



3.  $A \subseteq B \subseteq M$  Then  $(M/A)/(B/A) \cong M/B$
4.  $N \subseteq M$ . Then



preserves sums & intersections

10.3

R ring w/ unit.

Def  $M$  an  $R$ -module,  $N_1, N_2, \dots, N_k$  submodules of  $M$ .

1. The sum  $N_1 + N_2 + \dots + N_k = \{ n_1 + \dots + n_k \mid n_i \in N_i \}$   
is a submodule of  $M$ .

2. Let  $A \subseteq M$ . Then

$$RA = \{ r_1 a_1 + \dots + r_k a_k \mid r_i \in R, a_i \in A \}$$

is a submodule,

• the submodule generated by  $A$

• If  $N = RA$  we generate  $N$ .

3. A submodule is finitely generated if  $\exists$  a finite set which generates it. (possibly  $N = m$ )

4. A submodule is cyclic if it is generated by one element.  
i.e.  $N = Rn$  for some  $n \in M$ .

### Remarks

1.  $1 \in R$  ensures  $A \subseteq RA$ .

2.  $RA$  is always a submodule, smallest containing  $A$ .

3.  $N_1 + \dots + N_k$  is submodule generated by  $\cup N_i$ .

4. Submodule generation is generalization of "span" in a vector space.

5. Much easier than subgroups or subrings  
ideal gen by an element since action  
is all on the left.

## Examples

1.  $\mathbb{Z}$ -modules then cyclic module = cyclic group
2.  $R$  w/ 1 and  $M = {}_R R$ . Then  $M$  is cyclic generated by 1.
  - submodules  $\leftrightarrow$  left ideals
  - cyclic submodules  $\leftrightarrow$  principal ideals

## Warning

1. Submodules of f.g. module may not be finitely generated  
EX  $R = \langle x_1, x_2, x_3, \dots \rangle = M$  finitely generated  
 $N = \langle x_1, x_2, \dots \rangle$  Not f.g. submodule

## Define Direct Product / Direct Sum

Then Let  $N_1, \dots, N_k$  submodules of  $M$  TFAE

1.  $N_i \cap \sum_{j \neq i} N_j = 0$
2.  $N_i \cap (N_1 + \dots + N_k) = 0 \quad \forall i$
3. Each  $x \in N_1 + \dots + N_k$  is uniquely of form  $\sum x_i$

compare to subgroups

If so and  $M = N_1 + \dots + N_k$  then

$$M = N_1 \oplus \dots \oplus N_k$$