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1/23/07

<u>Recall</u> M, N R-module. A <u>module homomorphism</u> is a map $f: M \to N$ so that

- $f(m_1 + m_2) = f(m_1) + f(m_2) \quad \forall m_i \in M$
- $f(rm) = r f(m) \quad \forall r \in R, m \in M$

i.e.

$$\begin{tikzcd} m \arrow[r, "f"] \arrow[d, "r\cdot"'] & f(m) \arrow[d, "r\cdot"] \\ rm \arrow[r, "f"] & \begin{array}{c} rf(m) \\ f(rm) \end{array} \end{tikzcd}$$

"f commutes with action of R"

$Hom_R(M,N)$ is set of module homomorphisms from $M \to N$.
What structure does it have?

- $Hom_R(M,N)$ is an abelian group
- R commutative $\Rightarrow$ $Hom_R(M,N)$ is an R-module via
$$(rf)(m) = r \cdot f(m)$$
- $End_R(M) := Hom_R(M,M)$ is a ring w/ 1. It is a R-algebra if R is commutative

When R is a field then $End_F(M)$ are scalar transformations

<u>Quotient Modules</u>

Let $N \subseteq M$ be a submodule. Then M/N is an abelian group.

<u>Prop</u> M/N is an R-module via $r(m+N) := rm + N$

<u>Proof</u> Need to check well-defined!

In particular $\pi: M \to M/N$ is a module homomorphism w/ kernel N.

<u>Summary</u> Groups: Normal subgroups $\longleftrightarrow$ Kernel of a group homomorphism
Rings: Ideals $\longleftrightarrow$ Kernels of ring homomorphisms
R-modules: submodules $\longleftrightarrow$ Kernels of module homomorphisms

<u>Def.</u> Let N_1, N_2 be submodules of M. Then

$N_1 + N_2 = \{ n_1 + n_2 \mid n_i \in N_i \}$ is a submodule, sum of 2 submodules

<u>Rmk</u> Module homomorphisms are all homs of abelian groups, so $\cong$ thms hold. Just check the $\cong$'s are actually R-module maps

<u>Isomorphism Theorems</u> M, N R-modules

1. $\varphi : M \to N$ then $M/\ker\varphi \cong \varphi(M)$

2. Suppose A, B submodules of M. Then $(A+B)/B \cong A/A\cap B$

$$\begin{tikzcd}
 & A+B \arrow[dl, dash, "\cong"'] \arrow[dr, dash] & \\
B \arrow[dr, dash] & & A \arrow[dl, dash, "\cong"] \\
 & A\cap B &
\end{tikzcd}$$

3. $A \subseteq B \subseteq M$ Then $(M/A)/(B/A) \cong M/B$

4. $N \subseteq M$. Then

lattice of submodules of M/N $\longleftrightarrow$ lattice of submodules of M containing N.

preserves sums & intersections

10.3

R ring w/ unit.

Def M an R-module, $N_1, N_2, \ldots, N_k$ submodules of M.

1. The <u>sum</u> $N_1 + N_2 + \ldots + N_k = \{ n_1 + \ldots + n_k \mid n_i \in N_i \}$ is a submodule of M.

2. Let $A \subseteq M$. Then $RA = \{ r_1 a_1 + \ldots + r_k a_k \mid r_i \in R, a_i \in A \}$ is a submodule, <u>the submodule generated by A</u>
   - If $N = RA$ say A <u>generates</u> N.

3. A submodule is <u>finitely generated</u> if $\exists$ a finite set which generates it. (possibly $N = M$)

4. A submodule is <u>cyclic</u> if it is generated by one element i.e. $N = Rn$ for some $n \in N$.

<u>Remarks</u>

1. $1 \in R$ ensures $A \subseteq RA$
2. RA is always a submodule, smallest containing A
3. $N_1 + \ldots + N_k$ is submodule generated by $\cup N_i$
4. Submodule generation is generalization of "span" in a vector space
5. Much easier than subgroup or ~~subring~~ ideal gen by an element since action is all on the left.

Examples

1. $\mathbb{Z}$-modules then cyclic module = cyclic group
2. R w/ 1 and $M = {}_R R$. Then M is cyclic generated by 1

- submodules $\longleftrightarrow$ left ideals
- cyclic submodules $\longleftrightarrow$ principal ideals

Warning

1. Submodules of f.g. module may not be finitely generated

Ex $\quad R = F[x_1, x_2, x_3, \dots] = M$ finitely generated

$N = (x_1, x_2, \dots)$ Not f.g. submodule

Define Direct Product / Direct Sum

Thm Let $N_1, \dots, N_k$ submodules of M TFAE

1. $N_1 \times \dots \times N_k \cong N_1 + \dots + N_k$
2. $N_i \cap (N_1 + \dots + \hat{N_i} + \dots + N_k) = 0 \quad \forall i$
3. Each $x \in N_1 + \dots + N_k$ is uniquely of form $\sum n_i$

compare to subgroups

If so and $M = N_1 + \dots + N_k$ $\Rightarrow$

$M = N_1 \oplus \dots \oplus N_k$