1/30/07

Review $R \subset S$ a subring, N an R-module.

Def $S \otimes_R N$ is free abelian group on $S \times N \Big/ \underbrace{\left\langle \begin{array}{l} (s_1+s_2, n) = (s_1,n)+(s_2,n) \\ (s, n_1+n_2) = (s,n_1)+(s,n_2) \\ (sr, n) = (s, rn) \end{array} \right\rangle}_{H}$

Coset of $(s,n) = s \otimes n$

Thm/Def $S \otimes_R N$ is a left S-module via $s\left(\sum s_i \otimes n_i\right) = \sum (ss_i) \otimes n_i$

Proof

A Action is well defined, i.e. if $s_1 \otimes n_1 = s_2 \otimes n_2$ then $ss_1 \otimes n_1 = ss_2 \otimes n_2$

ETS H is preserved under action since

$$\begin{aligned} s_1 \otimes n_1 = s_2 \otimes n_2 &\longleftrightarrow s_1 \otimes n_1 - s_2 \otimes n_2 = 0 \\ &\longleftrightarrow (s_1,n_1) - (s_2,n_2) \in H \\ &\overset{?}{\longleftrightarrow} s((s_1,n_1)-(s_2,n_2)) \in H \\ &\longleftrightarrow ss_1 \otimes n_1 = ss_2 \otimes n_2 \end{aligned}$$

Apply to Σ's

But $s((s_1+s_2,n)-(s_1,n)-(s_2,n)) = ((ss_1+ss_2,n)-(ss_1,n)-(ss_2,n)) \in H$
etc..

Rest of axioms are easy.

$S \otimes_R N$ extending scalars from R to S

<u>Def</u> $i: N \to S\otimes_R N \quad i(n) = 1\otimes n$

$i(rn) = 1\otimes rn = r\otimes n = r(1\otimes n)$ so

i is an R-module homomorphism.

- i may not be 1-1, since the relations we imposed.
- we only imposed minimal relations to make S act.

<u>Universal Property.</u>

Let $R \subset S$, N a left R-module, $i: N \to S\otimes_R N$ as above

Suppose L is any left R-module and $\varphi: N \to L$ an R-module map

Then $\exists$ a unique S-module map $\tilde{\varphi}: S\otimes_R N \to L$ so

$\varphi = \tilde{\varphi}\circ i$, i.e.,

$$\begin{tikzcd} N \arrow[r, "i"] \arrow[dr, "\varphi"'] & S\otimes_R N \arrow[d, dashed, "\exists\,\tilde{\varphi}"] \\ & L \end{tikzcd}$$

Conversely if $\tilde{\varphi}: S\otimes_R N$ is an S-module map

then $\tilde{\varphi}\circ i$ is an R-module map

Proof

Suppose $\varphi: N \to L$ R-module map

Map $S \times N \to L$ by $(s,n) \mapsto s\varphi(n)$

This extends to a map on free group; check it maps to zero.

Thus we get

$$\tilde{\varphi}: \underset{\substack{\| \\ S \otimes_R N}}{F/H} \to L \text{ with } \tilde{\varphi}(s \otimes n) = s\varphi(n)$$

Now $s'\tilde{\varphi}(s \otimes n) = s's\varphi(n) = \tilde{\varphi}(ss' \otimes n)$ so $\tilde{\varphi}$ is S-map.

Thus $\tilde{\varphi}$ exists

Observe $\{1 \otimes n\}$ generates $S \otimes_R N$ so and

$\tilde{\varphi}(1 \otimes n) = \varphi(n)$ so done. //

COR $i: N \to S \otimes_R N$. Then $N/\ker i$ is largest quotient of N that embeds in an S-module.

In particular N is an R-submodule of some S-module iff i is 1-1!

Examples

1. $R\otimes_R N \cong N$

   Proof: Take $\varphi = id : N \to N$

$$\begin{tikzcd} N \arrow[r, "i"] \arrow[dr] & R\otimes_R N \arrow[d, "\Phi"] \\ & N \end{tikzcd}$$

   So $\Phi(r\otimes n) = \varphi(rn)$ so $\Phi = i^{-1}$.

2. $\mathbb{Q}\otimes_{\mathbb{Z}}$ Finite Abelian $= 0$, i.e. No finite abelian group embeds in a $\mathbb{Q}$-module.

3. Vector space extension of scalars

4. Induction in representation theory

General Tensor Products

Rmk $(sr, n) - (s, rn)$ requires only S a right R-module. Only used left structure on S to get S-module structure.

Def M_R right R-module, ${}_RN$ left R-module.

$$M\otimes_R N = \text{Free } \mathbb{Z}\text{-abelian group on } M\times N \Big/ \left\langle \begin{array}{l} (m_1+m_2, n) - (m_1, n) - (m_2, n) \\ (m, n_1+n_2) - (m, n_1) - (m, n_2) \\ (mr, n) - (m, rn) \end{array} \right.$$

etc..

Rmks

1. Many ways to write $m\otimes n$
2. Elements of $M\otimes N$ not all simple tensors.

3. Still have $i: M\times N \to M\otimes_R N$ $\quad i(m,n)=m\otimes n$

$$i(m_1+m_2,n) = i(m_1,n)+i(m_2,n)$$
$$i(m,n_1+n_2) = i(m,n_1)+i(m,n_2)$$
$$i(mr,n) = i(m,rn)$$

Define balanced (middle linear) map

$M\times N \to L$, L abelian group

Yet another universal property

Thm R, L, M_R, ${}_RN$, $i: M\times N \to M\otimes_R N$ as above.

1. Suppose $\Phi: M\otimes_R N \to L$ is a group hom. Then $\Phi\circ i: M\times N\to L$ is balanced

2. Suppose $\varphi: M\times N \to L$ is balanced. $\exists !$ group hom $\Phi: M\otimes_R N \to L$ s.t. $\varphi = \Phi\circ i$.