2/1/07

Review R-modules M_R, ${}_RN$ and abelian group L. $\varphi: M\times N \to L$ is *middle linear* if

$$\varphi(m_1+m_2, n) = \varphi(m_1,n) + \varphi(m_2,n)$$
$$\varphi(m, n_1+n_2) = \varphi(m,n_1) + \varphi(m,n_2)$$
$$\varphi(m, rn) = \varphi(mr, n) \qquad \forall\, m, m_i \in M,\ n, n_i \in N,\ r \in R$$

Key **Example** The map $i(m,n) = m\otimes n$ from $M\times N \hookrightarrow M\otimes_R N$ is middle linear by virtue of the relations imposed in the definition of $M\otimes_R N$.

Thm (Universal Property of tensor product) $(R,1)$, M_R, ${}_RN$, i, $M\otimes_R N$ as above.

1. Suppose $\Phi: M\otimes_R N \to L$ is a group homomorphism. Then $\Phi \circ i$ is balanced.

2. Suppose $\varphi: M\times N \to L$ is balanced. Then $\exists!\ \Phi: M\otimes_R N \to L$ such that $\varphi = \Phi\circ i$

$$\begin{tikzcd} M\otimes_R N \arrow[dr, dashed, "\exists\, \Phi"] & \\ M\times N \arrow[u, hook, "i"] \arrow[r, "\varphi"'] & L \end{tikzcd}$$

$$\begin{matrix} \text{Balanced maps} \\ M\times N \to L \end{matrix} \longleftrightarrow \begin{matrix} \text{Group homos} \\ M\otimes_R N \to L. \end{matrix}$$

Proof 1. $\Phi \circ i\ (m,n) = \Phi(m \otimes n) = \Phi(m \otimes rn) = \Phi \circ i\ (mr, n)$

$\Phi \circ i\ (m_1+m_2, n) = \Phi((m_1+m_2) \otimes n) = \Phi(m_1 \otimes n + m_2 \otimes n) = \Phi(m_1 \otimes n) + \Phi(m_2 \otimes n)$

$= \Phi \circ i(m_1, n) + \Phi \circ i(m_2, n)$

etc.

2. Let F be free abelian group on $M \times N$, recall that $M \otimes_R N$ is $F/$relations

Now φ extends (by UP of free ab groups) to a group homo $\hat{\varphi}: F \to L$. But φ balanced makes {relations} in kernel, so we get $\Phi: M \otimes_R N \to L$. Then

$\Phi(m \otimes n) = \hat{\varphi}(m,n) = \varphi(m,n)$ so $\varphi = \Phi \circ i$

But $\{m \otimes n\}$ generate $M \otimes_R N$ //

Application To define a group homo $M \otimes_R N \to L$ we must only define a balanced map $M \times N \to L$.

Cor D abelian, $i': M \times N \to D$ balanced such that

1. Image of i' generates D
2. Every balanced map on $M \times N$ factors through i'.

Then $\exists\ f: M \otimes_R N \cong D$ w/ $i' = f \circ i$.

Recall $R \subset S$, ${}_R M$ an R-module then $S \otimes_R M$ is a left S-module simp S acts on S on left.

Def S, R rings w/ 1, M an abelian group is an (S,R) bimodule if it is a left S module and a right R-module and $s(mr) = (sm)r$ $\forall s \in S, r \in R, m \in M$. Write ${}_S M_R$

Examples

1. Any ring R is an R-R bimodule (assoc of mult)
2. R commutative means any R-module is actually an R-R bimodule.
3. R a subring of S then S is an S-R bimodule

Suppose R is commutative, M, N R-modules. Then M is an R-R bimodule so $M \otimes_R N$ is a left R-module.

$$r(m \otimes n) = rm \otimes n = mr \otimes n = m \otimes rn$$

Notice that $r(i(m,n)) = i(rm,n) = i(m,rn)$

Def R comm w/1, M, N, L left R-module.
$\varphi: M \times N \to L$ is R-bilinear if ...

Cor As above then

$$\left\{ \begin{array}{c} R\text{-bilinear maps} \\ \varphi: M \times N \to L \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} R\text{-module homs} \\ \Phi: M \otimes_R N \to L \end{array} \right\}$$

Example

1. $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/3\mathbb{Z}$ $\quad a \otimes b = 3a \otimes b = a \otimes 3b$
$= a \otimes 0 = 0$

Thus $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/3\mathbb{Z} = 0$.

2. $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z}$

Prop $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/(m,n)\mathbb{Z}$

Proof $a \otimes b = ab(1 \otimes 1)$ so cyclic.
Also $m(1\otimes 1) = n(1 \otimes 1) = 0$ so order divides m, order $|n$
so order $|(m,n)$

Finally define $\varphi: \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/(m,n)\mathbb{Z}$ by
$\varphi(a,b) = ab$ is $\mathbb{Z}$-bilinear so induces
$\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/(m,n)\mathbb{Z}$ taking $1 \otimes 1 \mapsto 1$, so onto.

Ex Compare $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}$, $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ as left $\mathbb{C}$-modules

Elementary Properties of Tensor Products

1. Associative $({}_S M_R \otimes_R N_T) \otimes_T L \cong {}_S M_R \otimes_R (N_T \otimes_T L)$

For example let M, N, L be R-modules, R comm. Then
$(M \otimes N) \otimes L \cong M \otimes (N \otimes L)$ as left R-modules

Warning! Let M, N be R-modules, R commutative. Let $U \subset M$ be a submodule. Then
$U \otimes N$ and $M \otimes N$ are both R-modules
but it may not be that $M \otimes N$ has a submodule $\cong$ to $U \otimes N$.

Example $R = \mathbb{Z}$ $U = \mathbb{Z} \subset \mathbb{Q} = M$, N a finite abelian grp
$\mathbb{Z} \otimes_{\mathbb{Z}} N \cong N$
$\mathbb{Q} \otimes_{\mathbb{Z}} N = 0$.

2. $\oplus$ and $\otimes$ distribute by $+$ and $\times$. i.e.
M_R, M'_R, ${}_RN$, ${}_RN'$ then

$$(M \oplus M') \otimes_R N \cong (M \otimes_R N) \oplus (M' \otimes_R N)$$

$$M \otimes_R (N \oplus N') \cong (M \otimes_R N) \oplus (M \otimes_R N')$$

- If M, M' are left S-modules then these $\cong$s are of left S-modules.

3. R commutative, M, N left modules. Then

$$M \otimes_R N \cong N \otimes_R M \text{ as } R\text{-modules.}$$