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2/6/07

General Problem

Given two R-modules M, N does there exist an R-module U with $M \subset U$ and $U/M \cong N$?

extension problem, i.e. is there a SES

$$0 \to M \to U \to N \to 0 ?$$

Recall
- $M \xrightarrow{f} U \xrightarrow{g} W$ is exact at U if $\mathrm{Im} f = \ker g$
- Exact sequence means exact at each spot
- SES is $0 \to A \to B \to C \to 0$.

RMK Given $0 \to M \to U \to N \to 0$, U is not determined by M & N.

Example
$$0 \to \mathbb{Z}/p\mathbb{Z} \xrightarrow{1 \oplus 0} \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0$$

$$0 \to \mathbb{Z}/p\mathbb{Z} \longrightarrow \mathbb{Z}/p^2\mathbb{Z} \xrightarrow{\cdot p} \mathbb{Z}/p\mathbb{Z} \to 0$$

two different extensions of $\mathbb{Z}/p\mathbb{Z}$ by $\mathbb{Z}/p\mathbb{Z}$

Example 2 Group SES

$$1 \to Z_2 \xrightarrow{i} D_8 \to Z_2 \times Z_2 \to 1 \quad i(Z_2) = \{r^2, e\}$$
$$1 \to Z_2 \xrightarrow{i} Q_8 \to Z_2 \times Z_2 \to 1 \quad i(Z_2) = \{\pm 1\}$$

We need a notion of equivalence between short exact sequences

Equivalence & $\cong$ of Extensions

Def Let $0 \to M \to U \to N \to 0$ and $0 \to \hat{M} \to \hat{U} \to \hat{N} \to 0$ be SES of R-modules. A <u>homomorphism</u> of SES is a triple f, g, h so

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, "f"] & U \arrow[r] \arrow[d, "g"] & N \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & \hat{M} \arrow[r] & \hat{U} \arrow[r] & \hat{N} \arrow[r] & 0 \end{tikzcd}$$

commutes

It is an $\cong$ if all 3 maps are $\cong$s. U and $\hat{U}$ are <u>$\cong$ extensions</u>.

Def If $M = \hat{M}$, $N = \hat{N}$ and f & h are identity then $U, \hat{U}$ are <u>equivalent</u>.

equivalent $\Rightarrow \cong$.

<u>Example</u>

$$0 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{\psi} \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \xrightarrow{\varphi_i} \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \to 0$$

$\psi: 1 \mapsto (2, 0)$

$\varphi_1(a, b) = (a, b)$ $\quad \ker \varphi_1 = \{(0,0), (2,0)\} = \ker \varphi_2$

$\varphi_2(a, b) = (b, a)$

Lemma These are $\cong$ extensions:

Proof:

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r, "\psi"] \arrow[d, "id"] & \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \arrow[r, "\varphi_1"] \arrow[d, "id"] & \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \arrow[r] \arrow[d, "T"] & 0 \\ 0 \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r, "\psi"] & \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \arrow[r, "\varphi_2"] & \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \arrow[r] & 0 \end{tikzcd}$$

$$T(a,b) = (b,a)$$

Lemma They are not equivalent.

Proof Suppose f gives the equivalence

$$\varphi_1(0,1) = (0,1)$$

Thus $f(0,1)$ is either $(1,0)$ or $(3,0)$ both order 4, $\Rightarrow\Leftarrow$

Easy Five Lemma

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\psi"] \arrow[d, "\alpha"] & B \arrow[r, "\varphi"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & \tilde{A} \arrow[r, "\tilde{\psi}"] & \tilde{B} \arrow[r, "\tilde{\varphi}"] & \tilde{C} \arrow[r] & 0 \end{tikzcd}$$

1. α, γ 1-1 $\Rightarrow$ β 1-1
2. α, γ onto $\Rightarrow$ β onto
3. α, γ $\cong$ $\Rightarrow$ β $\cong$.

Proof 1. Let $b \in \ker \beta$. Thus $\gamma(\varphi(b)) = 0$ so $\varphi(b) = 0$ since γ 1-1

Thus $b = \psi(a)$. Thus $\tilde{\psi}(\alpha(a)) = 0$ so $\alpha(a) = 0$ so

$a = 0$ so $b = 0$. //

2. similar

3. obvious

Def $0 \to M \xrightarrow{f} U \xrightarrow{g} N \to 0$ is split if $\exists\ s: N \to U$ s.t.

$s \circ g = Id_N$.

Def Any map $s: N \to U$ with $g \circ s = id$ is a section of g.
When s is a module hom it is a splitting homomorphism

Prop Split iff $U \cong M \oplus \tilde{N}$ where $g: \tilde{N} \xrightarrow{\cong} N$.

Rmk Similarly for groups, split extensions are semidirect products.

Ex $1 \to \langle r \rangle \to D_8 \to \mathbb{Z}_2 \to 1$ split by $1 \to s$

$1 \to \langle i \rangle \to Q_8 \to \mathbb{Z}_2 \to 1$ nonsplit.

Prop For modules, $0 \to M \xrightarrow{f} U \xrightarrow{g} N \to 0$ is
split iff $\exists\ s: U \to M$ s.t. $s \circ f = id$

Hom Functors

Fix an R-module D and consider $\psi: M \to N$. Then

we get a natural map $\quad Hom_R(D, M) \xrightarrow{\tilde{\psi}} Hom_R(D, N)$

given by $\circ\, \psi$.

Lemma $\tilde{\psi}$ is a homo of abelian groups. If ψ is 1-1 so is $\tilde{\psi}$.

i.e. $\quad 0 \to M \to N$ gives

$0 \to Hom_R(D, M) \to Hom_R(D, N)$

A much harder question:

Suppose $\varphi: M \xrightarrow{\psi} N \to 0$ i.e.

$$0 \to \ker\psi \to M \xrightarrow{\psi} N \to 0.$$

Given $f \in \mathrm{Hom}_R(D, N)$ does it lift to a map $\tilde{f}: D \to M$

i.e. is

$$\mathrm{Hom}_R(D,M) \to \mathrm{Hom}_R(D,N) \quad \text{onto?}$$

Ex Not always

Ex $$0 \to \mathbb{Z} \xrightarrow{\cdot m} \mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \to 0$$

$D = \mathbb{Z}/m\mathbb{Z}$

$$\underset{\substack{\| \\ 0}}{\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z})} \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/m\mathbb{Z})$$

i.e.

Given $0 \to M \to U \to N \to 0$, D we get

$$(*) \quad 0 \to \mathrm{Hom}_R(D,M) \to \mathrm{Hom}_R(D,U) \to \mathrm{Hom}_R(D,N) \to 0$$

Not necess exact.

Prop If $0 \to M \to U \to N \to 0$ is split, so is $(*)$

Pf Hom distributes over $\oplus$.

<u>Theorem</u> P an R-module. TFAE

1. $0 \to L \to M \to N \to 0 \Rightarrow 0 \to \mathrm{Hom} \to \mathrm{Hom} \to \mathrm{Hom} \to 0$ exs

2. Given

$$\begin{tikzcd} & P \arrow[dl, dashed] \arrow[d, "f"] & \\ M \arrow[r, "\pi"] & N \arrow[r] & 0 \end{tikzcd}$$

$\exists\ \tilde{f} : P \to M$ sc
$\pi \circ \tilde{f} = f$.

3. Any $0 \to M \to N \to P \to 0$ splits

4. P is a direct summand of a free module

Say P is <u>projective</u>.