

2/8/07

Review  $M, N, D$   $R$ -modules. Suppose  $\Psi: M \rightarrow N$  a module homomorphism.  
We get a natural map

$$\text{Hom}_R(D, M) \xrightarrow{\tilde{\Psi}} \text{Hom}_R(D, N) \text{ given by} \\ f \mapsto \Psi \circ f.$$

- Prop
1.  $\tilde{\Psi}$  is a homomorphism of abelian groups.
  2.  $\Psi$  1-1  $\Rightarrow \tilde{\Psi}$  1-1, i.e.

$$0 \rightarrow M \rightarrow N \text{ induces } 0 \rightarrow \text{Hom}(D, M) \rightarrow \text{Hom}_R(D, N)$$

Proof

1.  $\tilde{\Psi}(f_1 + f_2) = \Psi \circ (f_1 + f_2) = \Psi \circ f_1 + \Psi \circ f_2 = \tilde{\Psi}(f_1) + \tilde{\Psi}(f_2)$
2. Suppose  $\Psi$  is 1-1 and  $\Psi \circ f_1 = \Psi \circ f_2$  then  $f_1 = f_2$ , so  $\tilde{\Psi}$  is 1-1

\*  $\text{Hom}_R(D, -)$  is called a left exact functor.

Harder Question

Suppose  $M \xrightarrow{\Psi} N \rightarrow 0$ , is

$$\tilde{\Psi}: \text{Hom}_R(D, M) \rightarrow \text{Hom}_R(D, N) \text{ onto?}$$

i.e. Given a map  $f: D \rightarrow N$ , does it lift to  $\tilde{f}: D \rightarrow M$ ?

$$\begin{array}{ccc} & D & \\ \swarrow & \downarrow f & \\ M & \xrightarrow{\Psi} & N \rightarrow 0 \end{array}$$

Partial answer: Not always!

Example  $0 \rightarrow \mathbb{Z} \xrightarrow{\cdot m} \mathbb{Z} \xrightarrow{\psi} \mathbb{Z}/m\mathbb{Z} \rightarrow 0$

Let  $D = \mathbb{Z}/m\mathbb{Z}$

$\text{Hom}_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}) \rightarrow \text{Hom}_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/m\mathbb{Z})$   
 not onto.

\*  $\text{Hom}_R(D, -)$  is not right exact.

Lemma  $\text{Hom}_R(A, B \oplus C) \cong \text{Hom}_R(A, B) \oplus \text{Hom}_R(A, C)$

$\text{Hom}_R(A \oplus B, C) \cong \text{Hom}_R(A, C) \oplus \text{Hom}_R(B, C)$

COR Suppose  $0 \rightarrow M \rightarrow U \rightarrow N \rightarrow 0$  is split. Then

$0 \rightarrow \text{Hom}_R(D, M) \rightarrow \text{Hom}_R(D, U) \rightarrow \text{Hom}_R(D, N) \rightarrow 0$

Proof  $U \cong M \oplus N$

Problem For which modules  $D$  is the functor  $\text{Hom}_R(D, -)$  exact?

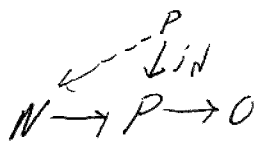
Theorem Let  $P$  be an  $R$ -module. TFAE

1. Given  $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$  then  $0 \rightarrow \text{Hom}_R(P, L) \rightarrow \text{Hom}_R(P, M) \rightarrow \text{Hom}_R(P, N) \rightarrow 0$ ,  
 i.e.  $\text{Hom}_R(P, -)$  is exact.
2. Given  $M \xrightarrow{\pi} N \rightarrow 0$ ,  $\exists \tilde{f}: P \rightarrow M \mid \pi \circ \tilde{f} = f$
3. Any SES  $0 \rightarrow M \rightarrow N \rightarrow P \rightarrow 0$  splits
4.  $P$  is a direct summand of a free module.

$P$  is called projective.

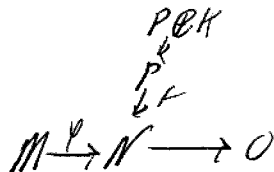
Proof 1  $\leftrightarrow$  2  $\checkmark$

2  $\rightarrow$  3



3  $\rightarrow$  4  $0 \rightarrow K \rightarrow \text{Free} \rightarrow P \rightarrow C$

4  $\rightarrow$  2 Suppose  $\mathcal{F}(S) = P \otimes K$ , given



Let  $s \in S$ . Consider  $f \otimes \pi(s) = \eta_s \in N$ .

Choose  $m_s$  s.t.  $\forall (m_s) = \eta_s$ .

Define  $\tilde{f}$  s.t.  $\tilde{f}(s) = m_s$  extends to ! map  $P \otimes K \rightarrow M$  //

Cor

1. Free  $\Rightarrow$  projective
2. Every module is  $\cong$  to a quotient of a projective module.
3.  $\text{Hom}_R(D, -)$  is exact iff  $D$  is projective.

Examples

1.  $R = F$  then all  $R$ -modules are free (Vector spaces have bases)
2. Finite abelian groups are never projective  $\mathbb{Z}$ -modules  
 - abelian groups w/ torsion are not projective
3. HW:  $\mathbb{Q}$  is not projective  $\mathbb{Z}$ -module.
4.  $R = \mathbb{Z}/6\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}$  ring  
 $P_1 = (1, 0)$   $P_2 = (0, 1)$  id sided idemp's  
 $R \cong P_1 \oplus P_2$  s.t. these are projective.

5. Direct sums and direct summands of projectives are projective

6.  $R = FG$  char  $F \neq 0$ .

structure of projectives very difficult.

## Injective Modules

Consider the functor  $\text{Hom}_R(-, D)$

Lemma  $M \rightarrow N \rightarrow 0$  gives  $0 \rightarrow \text{Hom}_R(N, D) \rightarrow \text{Hom}_R(M, D)$

Suppose  $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$

$0 \rightarrow \text{Hom}(N, D) \rightarrow \text{Hom}(M, D) \rightarrow \text{Hom}(L, D) \rightarrow 0$   
 $\uparrow$   
 may not be onto

i.e. every map  $L \rightarrow D$   
 does not necessarily lift to all of  $M$ .

Theorem Let  $Q$  be an  $R$ -module. TFAE

1. Given  $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$  exact then  $0 \rightarrow \text{Hom}_R(N, Q) \rightarrow \text{Hom}_R(M, Q) \rightarrow \text{Hom}_R(L, Q) \rightarrow 0$   
 is exact,

i.e.  $\text{Hom}_R(-, Q)$  is exact.

2. Given  $0 \rightarrow L \xrightarrow{i} M$   $\exists \hat{f}: M \rightarrow Q$  s.t.  $\hat{f} \circ i = f$   
 $\downarrow f$   
 $Q$  i.e. map lifts

3. Every short exact sequence  $0 \rightarrow Q \rightarrow M \rightarrow N \rightarrow 0$  splits

$Q$  is called injective.

Theorem Rwt 1, every module  $M$  is contained in an injective  $R$ -module

Sketching

$$0 \rightarrow L \xrightarrow{\psi} M \xrightarrow{\varphi} N \rightarrow 0 \quad R\text{-modules.}$$

$$\text{Suppose } D_R, \quad f: X \rightarrow Y \text{ gives } 1 \otimes f: D \otimes_R X \rightarrow D \otimes_R Y$$

$$d \otimes x \rightarrow d \otimes f(x)$$

i.e.  $D_R \otimes_R \_$  is covariant.

Example  $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Q}$

$$\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z} \rightarrow \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}$$

$$\Downarrow$$

Not l.t.e exact.

Prop/Def

1.  $D_R \otimes_R \_$  is right exact
2. It is exact iff only if  $D$  is flat.

Ex Projective  $\Rightarrow$  flat

Theorem (Adjoint Associativity) a.k.a. Change of Rings

$R, S$  rings  $A_R, {}_R B_S, C_S$  modules then

$$\text{Hom}_S (A \otimes_R B_S, C_S) \cong \text{Hom}_R (A_R, \text{Hom}_S ({}_R B_S, C_S))$$

as abelian groups. When  $R=S$  commutative this is an  $R$ -module map

$$\text{Hom}_R (A \otimes_R B, C) \cong \text{Hom}_R (A, \text{Hom}_R (B, C))$$

Rmk How is  $\text{Hom}_S ({}_R B_S, C_S)$  a right  $R$ -module. HW

Proof Let  $\varphi: A \otimes_R B \rightarrow C \in \text{LHS}$

Fix  $a \in A$ . Define

$$\Phi(a): B \rightarrow C \text{ by}$$

$$\Phi(a)(b) = \varphi(a \otimes b)$$

Conversely let  $\Phi: A \rightarrow \text{Hom}_S ({}_R B_S, C_S) \in \text{RHS}$

$$\text{Define } \varphi(a \otimes b) = \Phi(a)(b) \in C$$