p.403 ~~#2~~, 9, 10, 11, 37,

2/8/07

<u>Review</u> M, N, D R-modules. Suppose $\psi: M \to N$ a module homomorphism. We get a natural map

$$\mathrm{Hom}_R(D,M) \xrightarrow{\tilde{\psi}} \mathrm{Hom}_R(D,N) \quad \text{given by}$$
$$f \to \psi \circ f.$$

<u>Prop</u> 1. $\tilde{\psi}$ is a homomorphism of abelian groups.

2. ψ 1-1 $\Rightarrow$ $\tilde{\psi}$ 1-1, i.e.

$$0 \to M \to N \quad \text{induces} \quad 0 \to \mathrm{Hom}(D,M) \to \mathrm{Hom}_R(D,N)$$

<u>Proof</u>

1. $\tilde{\psi}(f_1+f_2) = \psi \circ (f_1+f_2) = \psi \circ f_1 + \psi \circ f_2 = \tilde{\psi}(f_1) + \tilde{\psi}(f_2)$!

2. Suppose ψ is 1-1 and $\psi \circ f_1 = \psi \circ f_2$. Then $f_1 = f_2$, so $\tilde{\psi}$ is 1-1.

* $\mathrm{Hom}_R(D,-)$ is called a <u>left</u> <u>exact</u> <u>functor</u>.

<u>Harder Question</u>

Suppose $M \xrightarrow{\psi} N \to 0$, is

$$\tilde{\psi}: \mathrm{Hom}_R(D,M) \to \mathrm{Hom}_R(D,N) \quad \text{onto?}$$

i.e. Given a map $f: D \to N$, does it lift to $\tilde{f}: D \to M$?

$$\begin{tikzcd} & D \arrow[dl, dashed] \arrow[d, "f"] & \\ M \arrow[r, "\psi"] & N \arrow[r] & 0 \end{tikzcd}$$

Partial answer: Not always!

Example $0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot m} \mathbb{Z} \xrightarrow{\psi} \mathbb{Z}/m\mathbb{Z} \longrightarrow 0$

Let $D = \mathbb{Z}/m\mathbb{Z}$

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/m\mathbb{Z})$$

not onto.

* $\mathrm{Hom}_R(D,-)$ is not right exact.

Lemma $\mathrm{Hom}_R(A, B \oplus C) \cong \mathrm{Hom}_R(A,B) \oplus \mathrm{Hom}_R(A,C)$

$\mathrm{Hom}_R(A \oplus B, C) \cong \mathrm{Hom}_R(A,B) \oplus \mathrm{Hom}_R(B,C)$

COR Suppose $0 \to M \to U \to N \to 0$ is split. Then

$$0 \to \mathrm{Hom}_R(D,M) \to \mathrm{Hom}_R(D,U) \to \mathrm{Hom}_R(D,N) \to 0$$

Proof $U \cong M \oplus N$ ✓

Problem For which modules D is the functor $\mathrm{Hom}_R(D,-)$ exact?

Theorem Let P be an R-module. TFAE

1. Given $0 \to L \to M \to N \to 0$ then $0 \to \mathrm{Hom}_R(P,L) \to \mathrm{Hom}_R(P,M) \to \mathrm{Hom}_R(P,N) \to 0$, i.e. $\mathrm{Hom}_R(P,-)$ is exact.

2. Given
$$\begin{array}{ccc} & P & \\ & \downarrow f & \\ M \xrightarrow{\pi} & N & \to 0 \end{array}$$
$\exists \tilde{f} : P \to M \mid f = \pi \circ \tilde{f}$

3. Any SES $0 \to M \to N \to P \to 0$ splits

4. P is a direct summand of a free module. P is called <u>projective</u>.

Proof. $1 \Leftrightarrow 2$ ✓

$2 \to 3$

$$\begin{tikzcd} & P \arrow[dl, dashed] \arrow[d, "id"] & \\ N \arrow[r] & P \arrow[r] & 0 \end{tikzcd}$$

$3 \to 4$ $\quad 0 \to K \to \text{Free} \to P \to 0$

$4 \to 2$ Suppose $\mathcal{F}(S) = P \oplus K$, given

$$\begin{tikzcd} & P \oplus K \arrow[d, "\pi"] & \\ & P \arrow[d, "f"] & \\ M \arrow[r, "\varphi"] & N \arrow[r] & 0 \end{tikzcd}$$

Let $s \in S$. Consider $f \circ \pi(s) = n_s \in N$.
Choose m_s s.t. $\varphi(m_s) = n_s$.
Define $\hat{f}$ s.t. $\hat{f}(s) = m_s$ extends to ! map $P \oplus K \to M$. //

Cor
1. Free $\Rightarrow$ projective
2. Every module is $\cong$ to a quotient of a projective module.
3. $\mathrm{Hom}_R(D, -)$ is exact iff D is projective.

Examples

1. $R = F$ then all R-modules are free (Vector spaces have bases)
2. Finite abelian groups are never projective $\mathbb{Z}$-modules
   - abelian groups w/ torsion are not projective
3. HW: $\mathbb{Q}$ is not projective $\mathbb{Z}$-module.
4. $R = \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$ ring

   $P_1 = (1,0)$ $\quad P_2 = (0,1)$ $\quad$ 2 sided ideals

   $R \cong P_1 \oplus P_2$ so these are projective.

5. Direct sums and direct summands of projectives are projective.

6. $R = FG$ char $F \mid |G|$.
structure of projectives very difficult.

Injective Modules

Consider the functor $Hom_R(-, D)$.

Lemma $M \to N \to 0$ gives $0 \to Hom_R(N, D) \to Hom_R(M, D)$

Suppose $0 \to L \to M \to N \to 0$

$$0 \to Hom(N, D) \to Hom(M, D) \to Hom(L, D) \to 0$$

may not be onto

i.e. every map $L \to D$
does not necessarily lift to all of M.

Theorem Let Q be an R-module. TFAE

1. Given $0 \to L \to M \to N \to 0$ exact then $0 \to Hom_R(N, Q) \to Hom_R(M, Q) \to Hom_R(L, Q) \to 0$ is exact,

i.e. $Hom_R(-, Q)$ is exact.

2. Given
$$\begin{array}{ccc} 0 \to L & \xrightarrow{i} & M \\ f\downarrow & & \\ Q & & \end{array}$$
$\exists\ \hat{f} : M \to Q$ s.t. $\hat{f} \circ i = f$

i.e. map lifts

3. Every short exact sequence $0 \to Q \to M \to N \to 0$ splits.

Q is called <u>injective</u>.

Theorem R w/ 1, every module M is contained in an injective R-module

Skinning

$$0 \to L \xrightarrow{\psi} M \xrightarrow{\varphi} N \to 0 \quad R\text{-modules.}$$

Suppose D_R, $f: X \to Y$ gives $1 \otimes f: D \otimes_R X \to D \otimes_R Y$

$$d \otimes x \to d \otimes f(x)$$

i.e. $D_R \otimes_R —$ is covariant.

Example $0 \to \mathbb{Z} \to \mathbb{Q}$

$$\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}$$

$$\shortparallel$$
$$0$$

Not left exact.

Prop/Def

1. $D_R \otimes_R —$ is right exact
2. It is exact if & only if D is flat.

Ex Projective $\Rightarrow$ flat

<u>Theorem</u> (Adjoint Associativity) a.k.a. Change of Rings

R, S rings A_R, ${}_R B_S$, C_S modules. Then

$$\mathrm{Hom}_S(A_R \otimes_R B_S, C_S) \cong \mathrm{Hom}_R(A_R, \mathrm{Hom}_S({}_R B_S, C_S))$$

as abelian groups. When $R = S$ is commutative this is an R-module map

$$\mathrm{Hom}_R(A \otimes_R B, C) \cong \mathrm{Hom}_R(A, \mathrm{Hom}(B, C))$$

<u>Rmk</u> How is $\mathrm{Hom}_S({}_R B_S, C_S)$ a right R-module. HW

<u>Proof</u> Let $\varphi : A \otimes_R B \to C \in LHS$

Fix $a \in A$. Define

$\Phi(a) : B \to C$ by

$\Phi(a)(b) = \varphi(a \otimes b)$

Conversely let $\Phi : A \to \mathrm{Hom}(B, C) \in RHS$

Define $\varphi(a \otimes b) = \Phi(a)(b) \in C$