

2/13/07

Recall An R -module P is projective iff it satisfies one of the following equivalent conditions:

1. $\text{Hom}_R(P, -)$ is an exact functor
2. Given any $M \xrightarrow{\varphi} N \rightarrow U$ and $f: P \rightarrow N$ then $\exists F: P \rightarrow M$ so $f = \varphi \circ F$.
3. Any SES $0 \rightarrow L \rightarrow M \rightarrow P \rightarrow 0$ splits, i.e. $M \cong L \oplus P$
4. \exists free R -module F so $F \cong K \oplus P$.

Now consider the contravariant functor $\text{Hom}_R(-, D)$

$$0 \rightarrow M \xrightarrow{i} U \xrightarrow{\pi} N \rightarrow 0 \quad \text{gives}$$

$$0 \rightarrow \text{Hom}_R(N, D) \xrightarrow{\star} \text{Hom}_R(U, D) \xrightarrow{\star} \text{Hom}_R(M, D)$$

Notice \star is \perp , if $f: N \rightarrow D$ is such that

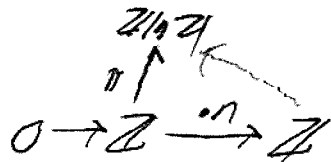
$f \circ \pi = 0$ then $f = 0$, i.e. Left exact

Question Is \star onto? I.E. Does a map from $M \rightarrow D$ lift to a map $U \rightarrow D$?

$$\begin{array}{c} P \text{ ?} \\ \uparrow \\ 0 \rightarrow M \rightarrow U \end{array}$$

A: Not always

EX $0 \rightarrow \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 0$



only possible map $\mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z}$ and its kernel is $2\mathbb{Z}$.

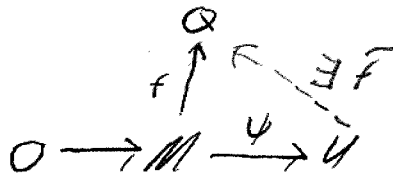
RMK As before if $0 \rightarrow M \rightarrow U \rightarrow N \rightarrow 0$ is split then SES

$$0 \rightarrow \text{Hom}_R(M, D) \rightarrow \text{Hom}_R(U, D) \rightarrow \text{Hom}_R(N, D) \rightarrow 0$$

Thm/Det Let Q be R -module, TFAE

1. $\text{Hom}_R(-, Q)$ is exact functor.

2. Given injection $0 \rightarrow M \xrightarrow{\psi} U$ and a map $f: M \rightarrow Q$ then \exists $\hat{f}: U \rightarrow Q$ so $\hat{f} \circ \psi = f$.



3. Any short exact sequence

$$0 \rightarrow Q \rightarrow U \rightarrow N \rightarrow 0 \text{ splits}$$

RMK 1. Q a submodule \rightarrow direct summand

a. Arrows reversed

3. Missing condition $\neq 4$!

Q is called injective

Summary

1. $\text{Hom}_R(-, D)$ and $\text{Hom}_R(D, -)$ are left exact functors from R -modules to abelian groups
2. $\text{Hom}_R(-, D)$ is contravariant, $\text{Hom}_R(D, -)$ is covariant
3. $\text{Hom}_R(D, -)$ is exact iff D is projective
4. $\text{Hom}_R(-, D)$ is exact iff D is injective

Prop Let Q be an R -module.

(1) (Baer's Criterion) Q is injective iff for every left ideal

I of R and any R -module homo $g: I \rightarrow Q$, g extends to $\tilde{g}: R \rightarrow Q$.

(2) R a P.I.D. Then Q is injective iff $rQ = Q \forall r \neq 0$.

In particular quotients of injectives are injective

Proof See book

Special Case $R = \mathbb{Z}$

Called divisible abelian group because you can "divide" by n .

Ex \mathbb{Q} , \mathbb{Q}/\mathbb{Z} are injective \mathbb{Z} -modules

Thm R.1. Every R -module is contained in an injective module.

Torsionary S.E.S.

Suppose $0 \rightarrow M \xrightarrow{i} U \xrightarrow{\pi} N \rightarrow 0$

Suppose D a right R -module. Given $f: A \rightarrow B$ Defn

$$1 \otimes f: D \otimes_R A \rightarrow D \otimes_R B \text{ by}$$

$$d \otimes a \rightarrow d \otimes f(a).$$

RAK $D \otimes_R -$ is covariant functor R -modules \rightarrow abelian groups
for to left S -modules if D is an
 (S, R) -bimodule.

PROP $D \otimes_R -$ is right exact.

DEF D is flat if $D \otimes_R -$ is exact.

EX PROJ \Rightarrow flat.

(3)

Change of Rings a.k.a. Adjoint Associativity

Two rings w/ modules ${}_R B_S, A_R, C_S$

Then

$$\text{Hom}_S(A \otimes_R B, C) \cong \text{Hom}_R(A, \text{Hom}_S(B, C))$$

as abelian groups

a. Hom as right modules

Remark 1. If $R=S$ commutative then \cong as R -modules

2. Why is $\text{Hom}_S({}_R B, C)$ a right R -module?

$$\text{Define } (fr)(b) = f(rb)$$

$$(f(r_1 r_2))(b) = f(r_1(r_2 b))$$

$$((f(r_1) r_2)(b)) = (f(r_1)(r_2 b)) = f(r_1(r_2 b)) \checkmark$$

3. Corollary R -commutative then tensor product of two projectives is projective.

PROOF LHS $\xrightarrow{\psi}$ RHS

Given $f: A \otimes B \rightarrow C$ and $a \in A$.

Define $\psi f a: B \rightarrow C$ by $\psi f a(b) = f(a \otimes b)$

RHS \rightarrow LHS

Given map $A \xrightarrow{g} \text{Hom}(B, C)$

Define map $A \otimes B \rightarrow C$ by

$$a \otimes b \rightarrow g(a)(b) \quad //$$