2/13/07

<u>Recall</u> An R-module P is <u>projective</u> iff it satisfies one of the following equivalent conditions:

1. $Hom_R(P,-)$ is an exact functor

2. Given any $M \xrightarrow{\varphi} N \to 0$ and $f: P \to N$ then $\exists$ $F: P \to M$ so $f = \varphi \circ F$.

3. Any SES $0 \to L \to M \to P \to 0$ splits i.e. $M \cong L \oplus P$

4. $\exists$ free R-module F so $F \cong K \oplus P$.

Now consider the contravariant functor $Hom_R(-, D)$

$$0 \to M \xrightarrow{i} U \xrightarrow{\pi} N \to 0 \quad \text{gives}$$

$$0 \to Hom_R(N,D) \xrightarrow{*} Hom_R(U,D) \xrightarrow{\star} Hom_R(M,D)$$

Notice $*$ is 1-1, if $f: N \to D$ is such that $f \circ \pi = 0$ then $f = 0$, i.e. $\boxed{\text{Left exact}}$

<u>Question</u> Is $\star$ onto? I.E. Does a map from $M \to D$ lift to a map $U \to D$?

$$\begin{tikzcd} & D & \\ 0 \arrow[r] & M \arrow[u] \arrow[r] & U \arrow[ul, dashed, "?"'] \end{tikzcd}$$

A: Not always

Ex $$0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$$

$$\begin{tikzcd}
 & \mathbb{Z}/n\mathbb{Z} & \\
0 \arrow[r] & \mathbb{Z} \arrow[u, "\pi"] \arrow[r, "\cdot n"] & \mathbb{Z} \arrow[ul, dashed]
\end{tikzcd}$$

only 1 possible map $\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ and its kernel is $n\mathbb{Z}$.

RMK As before if $0 \to M \to U \to N \to 0$ is split then SES

$$0 \to \mathrm{Hom}_R(M,D) \to \mathrm{Hom}_R(U,D) \to \mathrm{Hom}_R(M,D) \to 0$$

Thm/Def Let Q be R-module. TFAE

1. $\mathrm{Hom}_R(-,Q)$ is exact functor.

2. Given injection $0 \to M \xrightarrow{\psi} U$ and a map $f: M \to Q$ then $\exists$ $\hat{f}: U \to Q$ so $\hat{f} \circ \psi = f$.

$$\begin{tikzcd}
 & Q & \\
0 \arrow[r] & M \arrow[u, "f"] \arrow[r, "\psi"] & U \arrow[ul, dashed, "\exists \hat{f}"']
\end{tikzcd}$$

3. Any short exact sequence
$$0 \to Q \to U \to N \to 0 \text{ splits}$$

RMK 1. Q a submodule $\longrightarrow$ direct summand
2. Arrows reversed
3. Missing condition #4!

Q is called <u>injective</u>

Summary

1. $\mathrm{Hom}_R(-,D)$ and $\mathrm{Hom}_R(D,-)$ are left exact functors from R-modules to abelian groups
2. $\mathrm{Hom}_R(-,D)$ is contravariant, $\mathrm{Hom}_R(D,-)$ is covariant
3. $\mathrm{Hom}_R(D,-)$ is exact iff D is projective
4. $\mathrm{Hom}_R(-,D)$ is exact iff D is injective

Prop Let Q be an R-module.

(1) (Baer's Criterion) Q is injective iff for every left ideal I of R and any R-module homo $g: I \to Q$, g extends to $\tilde{g}: R \to Q$.

(2) R a P.I.D. Then Q is injective iff $rQ = Q$ $\forall r \neq 0$. In particular quotients of injectives are injective

Proof See book

Special Case $R = \mathbb{Z}$

Called divisible abelian group because you can "divide" by n.

Ex $\mathbb{Q}$, $\mathbb{Q}/\mathbb{Z}$ are injective $\mathbb{Z}$-modules

Thm R.1. Every R-module is contained in an injective module.

<u>Tensoring S.E.S.</u>

Suppose $0 \to {}_R M \xrightarrow{i} {}_R U \xrightarrow{\pi} {}_R N \to 0$

Suppose D a right R-module. Given $f : {}_R A \to {}_R B$ Define

$$1 \otimes f : D \otimes_R A \to D \otimes_R B \text{ by}$$

$$d \otimes a \to d \otimes f(a).$$

<u>Rmk</u> $D \otimes_R -$ is covariant functor R-modules $\to$ abelian groups (or to left S-modules if D is an (S,R)-bimodule

<u>Prop</u> $D \otimes_R -$ is <u>right exact</u>.

<u>DEF</u> D_R is <u>flat</u> if $D_R \otimes_R -$ is exact.

<u>Ex</u> Proj $\Rightarrow$ flat.

Change of Rings a.k.a. Adjoint Associativity

Thm R, S rings w/ modules ${}_R B_S$, A_R, C_S.

Then

$$\mathrm{Hom}_S(A_R \otimes_R B_S, C_S) \cong \mathrm{Hom}_R(A_R, \mathrm{Hom}_S({}_R B_S, C_S))$$

as abelian groups

0. Homs as right modules

Remarks 1. If $R = S$ commutative then $\cong$ as R-modules

2. Why is $\mathrm{Hom}_S({}_R B_S, C_S)$ a right R-module?

Define $(fr)(b) = f(rb)$

$(f(r_1 r_2))b = f(r_1 r_2 b)$

$((fr_1)r_2)(b) = (fr_1)(r_2 b) = f(r_1 r_2)b$ ✓

3. Corollary R-commutative then tensor product of two projectives is projective.

Proof: $LHS \xrightarrow{\Psi} RHS$

Given $f: A \otimes B \to C$ and $a \in A$.

Define $\Psi f(a): B \to C$ by $\Psi f(a)(b) = f(a \otimes b)$

$RHS \to LHS$

Given map $A \xrightarrow{g} \mathrm{Hom}(B, C)$

Define map $A \otimes B \to C$ by

$a \otimes b \to g(a)(b)$. //