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2/22/07

Basic Linear Algebra to Review on Own

(Mostly 11.1–11.2) or any lin. alg. text

- linear independence
- basis, existence of
- finite spanning set contains a basis
- dimension of a vector space, subspaces
- In f.d. v.s any finite lin. ind. set extends to a basis (also ∞ dim)
- Any n-dim V.S/F is $\cong F^n$
- Rank-nullity thm for linear transformations
- Matrix of a linear transf.
- Change of basis matrix, similar matrices
- Equiv. conditions of invertibility of a matrix
- Solving system of linear equations
- Determine basis for Kernel, image of a linear map

Dual Vector Spaces

Def Let V be a vector space $/F$. The dual space is

$$V^* = \mathrm{Hom}_F(V, F) \quad \text{is linear functionals.}$$

Def Suppose $\{v_1, v_2, \ldots, v_n\}$ a basis of V. Define
$v_i^* \in V^*$ by $\quad v_i^*(v_j) = \delta_i^j$

Thm $\{v_i^*\}$ is a basis of V. In particular $\dim V = \dim V^*$.
Called dual basis

Rmks 1. When $\dim V = \infty$ then $\dim V < \dim V^*$
2. Sometimes require continuous dual, etc.

Theorem There is a natural injection $V \overset{E}{\hookrightarrow} V^{**}$.
When $\dim V < \infty$ this is, of course, an $\cong$.

Proof Let $v \in V$. Define $E_v(f) = f(v) \in F \quad \forall f \in V^*$.
E_v = evaluation at v

1. $E_v(f_1 + cf_2) = E_v(f_1) + cE_v(f_2)$ so $E_v \in \mathrm{Hom}(V^*, F)$
2. $E_{v_1 + cv_2} = E_{v_1} + cE_{v_2}$ so $E : V \to V^{**}$ is linear.
3. Suppose $v \in \mathrm{Ker}\,E$. Thus $v \neq 0$ Extend v to a basis,
define $f \in V^*$ by $f(v) = 1$, $f = 0$ on other basis.
Then $E_v f = f(v) = 1$. Thus $E_v \neq 0$ $\cancel{=}$.

Rmk Thus for f.d. VS we have a natural $\cong$ $V \cong V^{**}$ which does not require choosing a basis. To show $V \cong V^*$ one needs a basis.

Def Let $\varphi : V \to W$ be a linear map. We get

$$\varphi^* : W^* \to V^* \text{ by } \quad \varphi^* f = f \circ \varphi \in V^* \text{ for } f \in W^*.$$

check φ^* is linear.

Thm Suppose V, W f.d. w/ bases chosen and dual bases for V^*, W^*. Then the matrix of φ^* is the transpose of that for φ.

Cor For a matrix A, Row rank = col rank.

Proof Let $\varphi : V \to W$ a linear map w/ matrix A.
Col Rank of A is $\dim(\operatorname{Im}\varphi)$
Row Rank of A = col rank A^t = $\dim \operatorname{Im} \varphi^*$

Let $f \in W^*$.

$$f \in \operatorname{Ker}\varphi^* \longleftrightarrow \underset{f\circ\varphi}{\cancel{\varphi \circ f}} = 0 \longleftrightarrow \underset{f(\varphi(v)) = 0}{\cancel{\varphi(f(v))}} = 0 \ \forall v \in V$$
$$\longleftrightarrow \varphi(V) \subseteq \operatorname{Ker} f$$
$$\longleftrightarrow f \in \operatorname{Ann}(\varphi(V))$$

Exercise $\dim \operatorname{Ann}(\varphi(V)) = \dim W - \dim \varphi(V)$
$\dim \operatorname{Ker}\varphi^* = \dim W^* - \dim \varphi^*(W^*)$

Thus $\dim \varphi(V) = \dim \varphi^*(W^*)$ //

Determinants

For this section assume R comm w/ id, $W, V_1, V_2, \ldots, V_n$ R-module.

Def. $\varphi : V_1 \times \cdots \times V_n \to W$ is *multilinear* if it is R-linear in each coordinate.

- If all $V_i = V$ then φ is n-multilinear of V
- If all $V_i = V$ and $W = R$, φ is n-multilinear *form*.

Example

1. Linear functionals
2. Bilinear forms

Def An n-multilinear form on V is *alternating* if

$\varphi(v_1, v_2, \ldots, v_n) = 0$ whenever $v_i = v_{i+1}$ some i.

It is *symmetric* if $\varphi(v_1, \ldots, v_n) = \varphi(v_{\sigma 1}, \ldots, v_{\sigma n})$ $\forall \sigma \in S_n$

Properties of Multilinear, alternating functions

1. $\varphi(v_1, \ldots, v_i, v_{i+1}, \ldots, v_n) = -\varphi(v_1, \ldots, v_{i+1}, v_i, v_{i+1}, \ldots, v_n)$
2. $\varphi(v_{\sigma 1}, \ldots, v_{\sigma n}) = \operatorname{sgn}(\sigma)\, \varphi(v_1, \ldots, v_n)$
3. If any $v_i = v_j$, $i \neq j$ then $\varphi(v_1, \ldots, v_n) = 0$
4. Replacing v_i by $v_i + \alpha v_j$ does not change φ

Proof ...

COR Let φ be n-multilinear, alternating on V. Suppose $\exists$ $v_1, \ldots, v_n$ and $w_1, \ldots, w_n \in V$ s.t.

$$w_1 = \alpha_{11} v_1 + \ldots + \alpha_{n1} v_n$$

$$\vdots$$

$$w_n = \alpha_{1n} v_1 + \ldots + \alpha_{nn} v_n$$

Then

$$\varphi(w_1, \ldots, w_n) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, \alpha_{\sigma(1)1}\, \alpha_{\sigma(2)2} \cdots \alpha_{\sigma(n)n}\, \varphi(v_1, \ldots, v_n)$$

Proof EXPAND!

DEF An nxn determinant function of R is a

$$\det : M_{n \times n}(R) \to R \quad \text{s.t.}$$

1. det is n-multilinear on R^n (tuple of columns)
2. $\det(Id) = 1$

Thm $\exists$ a unique nxn det function!