2/27/07

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Recall $R$ comm ring w 1, $V$ an $R$-module. An $(n)$-multilinear form is a function

$$\varphi: V \times V \times \cdots \times V \longrightarrow R \text{ such that } \varphi$$

is $R$-linear in each coordinate.

$\varphi$ alternating if repeat $\rightarrow \varphi = 0 \Rightarrow \varphi(v_{\sigma 1}, \ldots, v_{\sigma n}) = \operatorname{sgn}\sigma \, \varphi(v_1, \ldots, v_n)$

$\varphi$ symmetric if $\varphi(v_{\sigma 1}, \ldots, v_{\sigma n}) = \varphi(v_1, \ldots, v_n)$ $\forall \sigma \in S_n$.

Def An $n \times n$ determinant $\det: M_{n\times n}(R) \rightarrow R$ s.t.

1. multilinear, alternating form ($V = R^n$ columns)
2. $\det I = 1$

Theorem $\exists$ a ! $n \times n$ det on $R$ given by

$$\det A = \sum_{\sigma \in S_n} \operatorname{sgn}\sigma \; a_{\sigma(1)1} a_{\sigma(2)2} \cdots a_{\sigma(n)n}$$

Proof

1. Check $\det A$ as above works

2. Given $A$, notice $\text{col } 1 = a_{11}e_1 + a_{21}e_2 + \cdots + a_{n1}e_n$
$\text{col } 2 = \ldots$
$\text{col } n = a_{1n}e_1 + \cdots + a_{nn}e_n$

$$\det A = \sum_{\sigma \in S_n} \operatorname{sgn}\sigma \; a_{\sigma(1)1} \cdots a_{\sigma(n)n} \underbrace{\det(e_1, e_2, \ldots, e_n)}_{1}$$

Theorem (Properties of det)

1. Every alternating $R$-multilinear form on $Mat_n R$ is some multiple of det.

2. $\det(AB) = \det A \det B$

3. $A$ is invertible in $M_{n\times n}(R)$ iff $\det A$ is a unit in $R$ and $\det(A^{-1}) = \det A^{-1}$

4. $\det(A^t) = \det A$, in particular det is multilinear, alt on rows

5. Usual Row, column operations

~~Theorem (Cofactor expansion)~~

Def Let $A_{ij} = A$ w/ row $i$, col $j$ deleted.
$\det(A_{ij}) = ij$ minor
$(-1)^{i+j} \det A_{ij}$ is cofactor.

Thm $$\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij} \qquad \text{row } i$$

$$\det A = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij} \qquad \text{col } j.$$

"Proof" Show RHS is also alt, $n$-mult form

**Def.** Fix $A$. Define a new matrix

$A^{adj}$ by $A^{adj} = (b_{ij})$ $b_{ij} = (-1)^{i+j} \det A_{ji}$.

classical adjoint.

Ex $A = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 2 & 5 \\ 8 & -1 & -2 \end{pmatrix}$ minors $\begin{pmatrix} 3 & -40 & -16 \\ -5 & -10 & -25 \\ 13 & 5 & 2 \end{pmatrix}$

cofactors $A^{adj} = \begin{pmatrix} 3 & 5 & 13 \\ 40 & -10 & -5 \\ -16 & 25 & 2 \end{pmatrix}$

**Theorem** 1. $AA^{adj} = A^{adj}A = \det A \cdot I$

2. $A$ inv iff $\det A$ unit
and then $A^{-1} = \frac{1}{\det A} A^{adj}$

## Tensor Algebras

Setting: $R$ comm w/ 1, $M$ an $R$-module. Thus $M \otimes_R M$ is an $R$-module.

Def. $\mathcal{T}^0(M) = R$

$\mathcal{T}^k(M) = M \otimes_R M \otimes_R \cdots \otimes_R M$

$\mathcal{T}(M) = \bigoplus_{k=0}^{\infty} \mathcal{T}^k(M)$ tensor algebra of $M$.

Rmk 1. Elements of $\mathcal{T}(M)$ may look like

$$r + m + m_1 \otimes m_2 + m_3 \otimes m_4 \otimes m_5 \text{ etc.}$$

2. $M \cong \mathcal{T}^1(M)$ submodule

3. What is algebra structure?

Thm 1. $\mathcal{T}(M)$ is an $R$-algebra and is graded, i.e. $\mathcal{T}^i(M)\mathcal{T}^j(M) \subseteq \mathcal{T}^{i+j}(M)$.

2. Suppose $A$ is an $R$-algebra, $\varphi: M \to A$ an $R$-module map. Then $\exists! \ \Phi: T(M) \to A$.

Proof 1. Bilinear $\mathcal{T}^i(M) \times \mathcal{T}^j(M) \to \mathcal{T}^{i+j}(M)$ so mult well defined.

2. $(m_1, \ldots, m_k) \mapsto \varphi(m_1) \cdots \varphi(m_k)$ $R$-multilinear so we get $\Phi: \mathcal{T}^k(M) \to A$.

Example 1. $V$ a vector space / field $K$, f.d. $n$. Then

$\mathcal{T}(V) \cong$ poly ring in $n$ noncomm variables.

2. $R = \mathbb{Z}$, $M = \mathbb{Q}/\mathbb{Z}$ $\quad \mathcal{T}(M) = \mathbb{Z} \oplus \mathbb{Q}/\mathbb{Z}$

mult?

3. $R = \mathbb{Z}$, $M = \mathbb{Z}/n\mathbb{Z}$ $\quad M \otimes_{\mathbb{Z}} M \cong M$ so

$$\mathcal{T}(M) \cong \mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z} \oplus \cdots$$

$$\cong \mathbb{Z}[x]/(nx)$$

GRADED RINGS

1. Def
2. Homog els
3. Graded ideal $\leftarrow$ earlier HW.
4. Homs of graded rings

PROP Graded Ring / Graded ideal $\cong$ Graded Ring