HW p. 454 1, 2, 6, 13

3/1/07

Recall R comm w 1, M an R-module, $\mathcal{T}^k(M) = M \otimes_R M \otimes_R \cdots \otimes_R M$.

Def $\mathcal{T}(M) = \bigoplus_{i=0}^{\infty} \mathcal{T}^i(M) = R \oplus M \oplus (M \otimes_R M) \oplus \cdots$

tensor algebra, it is an R-algebra w/ a copy of M as an R-submodule
It is also a graded ring

Def A ring S is graded if $S = S_0 \oplus S_1 \oplus \cdots$ as abelian groups and $S_i S_j \subseteq S_{i+j}$. Elements of S_k are homogeneous of degree k.

Def An ideal $I \subseteq S$ is graded if $I = \bigoplus_{k=0}^{\infty} (I \cap S_k)$
A ring homomorphism between graded rings S & T is graded if $\varphi(S_k) \subseteq T_k \ \forall k$

Rmks
1. S_0 is always a subring
2. I is graded means that if $i = i_{k_1} + i_{k_2} + \cdots + i_{k_t}$ w/ $i_{k_j} \in S_k$ then each $i_{k_j} \in I$.

Examples 1) $S = R[x_1, \dots, x_n]$ is graded. S_k = homogeneous of degree k.
e.g. $R[x,y,z]$ $xy^3 + xy^2z \in S_4$

(x,y) is graded ideal
$(1+x)$ is NOT, $1+x \in (1+x)$ but $1 \notin (1+x)$

Prop I graded ideal then $S/I \cong \bigoplus_{k=0}^{\infty} S_k/I_k$

Symmetric Algebras

- $\mathcal{T}(M)$ has M as a submodule w/ no "extra" relations
  Suppose we also want our ring to be commutative?

Def: The symmetric algebra
$$\mathrm{Sym}(M) \cong \mathcal{T}(M) / \underbrace{\langle m_1 \otimes m_2 - m_2 \otimes m_1 \mid m_i \in M \rangle}_{\text{ideal generated}} = \mathcal{C}(M)$$

Rmk This is a commutative ring. $\mathcal{C}(M)$ gen by elts of degree 2.
Thus $\mathrm{Sym}(M) \cong \bigoplus \mathcal{T}(M)/\mathcal{C}_k(M)$

Def. $\mathrm{Sym}^k(M) = k^{th}$ symmetric power $\cong M \otimes_R M \otimes \cdots \otimes_R M / \langle m_1 \otimes \cdots \otimes m_k - m_{\sigma(1)} \otimes \cdots \otimes m_{\sigma(k)} \rangle$

like $\otimes$ but order doesn't matter.

Rmk 1. $\mathrm{Sym}(M)$ has universal property w.r.t. commutative R-algs.

2. $\mathrm{Sym}(V) \cong$ poly ring in $\dim V$ variables

Exterior Algebras

Def. The exterior algebra $\Lambda(M) \cong \mathcal{T}(M) / A(M)$

where $A(M)$ is ideal generated by all tensors of form $m \otimes m$.

Rmks

1. $\Lambda(M)$ is graded, $\Lambda(M) \cong \bigoplus_{i=0}^{n} \Lambda^i(M)$

   $\Lambda^0(M) = R$, $\Lambda^1(M) = M$ $\qquad \Lambda^k(M) =$ kth <u>exterior power</u>

2. $(x+y)\otimes(x+y) \equiv 0 \Rightarrow x\otimes y = -y\otimes x$

3. Multiplication usually written w/ a wedge, i.e.

   $m_1\otimes\cdots\otimes m_k + A(M) \in \Lambda(M)$ is denoted

   $m_1\wedge\cdots\wedge m_k$.

4. Thus $m_1\wedge\cdots\wedge m_k = \operatorname{sgn}(\sigma)\, m_{\sigma(1)}\wedge\cdots\wedge m_{\sigma(k)}$

   <u>WARNING</u> ~~$a\wedge b$ may not be $b\wedge a$~~

   ab may not be $-ba$ for $b,a \in \Lambda(M)$.

5. Universal property

$$\begin{tikzcd} & \Lambda^k(M) \arrow[dr, dashed, "\tilde{\varphi}"] & \\ M\times\cdots\times M \arrow[ur, "i"] \arrow[rr] & & N \end{tikzcd}$$

$\varphi: M\times\cdots\times M \to N$ alternating multilinear

then $\exists!\ \tilde{\varphi}: \Lambda^k(M) \to N$.

Example

Let $R = k$, $M = k^n = \langle e_1, \ldots, e_n \rangle$

Determine $\Lambda^s(M)$.

Homomorphisms

Suppose $\varphi : M \to N$ an R-module homo. We get

$$\tau^k(\varphi) : \tau^k(M) \longrightarrow \tau^k(N)$$

which gives

$$S^k_{ym}(\varphi) : S^k_{ym}(M) \longrightarrow S^k_{ym}(N)$$
$$\Lambda^k(\varphi) : \Lambda^k(M) \to \Lambda^k(N)$$

Special case V an n-dim vector space
$\varphi : V \to V$ linear.

$\Lambda^n(\varphi) : \Lambda^n(V) \to \Lambda^n(V)$ is mult by a scalar.

Theorem This scalar is $\det \varphi$.

Proof Pick a basis $v_1, \ldots, v_n$.

$$\Lambda^n(\varphi) : v_1 \wedge \ldots \wedge v_n = \varphi(v_1) \wedge \ldots \wedge \varphi(v_n)$$
$$= \left(\sum a_{i1} v_i\right) \wedge \sum a_{i2} v_i \wedge \ldots \wedge$$

$\vdots$