p 468 #1, 2, 5, 13

3/13/07 Modules over PIDs

Some finiteness conditions.. R a ring, M an R-module.

Def M is Noetherian if it satisfies the ascending chain condition (ACC), i.e.

$$M_1 \subseteq M_2 \subseteq \cdots \subseteq M \Rightarrow \exists m \mid M_m = M_{m+1} = \cdots$$

R is Noetherian if ${}_R R$ is

Thm TFAE

1. M is Noetherian
2. Every nonempty set of submodules of M contains a maximal element under inclusion.
3. Every submodule of M is f.g.

Proof 1→2 easy 2→3 Choose $\Sigma = \{$ f.g. submodules of $M \}$ choose max elt 3→1 easy

Examples

1. R Noeth $\Rightarrow R[x]$ is.
2. $F[x_1, x_2, \ldots]$ not Noeth.

* 3. Any PID is clearly Noeth by condition #3

<u>Prop</u> Let R be an int. dom. Let $M \cong R^n$ be free of rank n. Then any $n+1$ elts of M are R-linearly depen.

<u>Pf</u> Embed R in quotient field, use linear alg.

<u>Def</u> Let R be I.D., M an R-module. The <u>rank</u> of M is the maximal # of R-linearly indep elts of M.

<u>Ex</u> $R = \mathbb{Z}$, ~~both~~ $\mathbb{Z} \oplus \mathbb{Z}$ and $\mathbb{Z}/_{5\mathbb{Z}} \oplus \mathbb{Z}$
~~have rank 2~~ rank 2 rank 1

<u>Key Theorem</u> R a PID, M free of rank n, N submodule. Then:

1. N is free of rank $m \leq n$

2. $\exists$ a basis $y_1, y_2, \ldots, y_n$ of M and nonzero elts $a_1, a_2, \ldots, a_m$ of R such that
$$a_1 | a_2 | a_3 | \cdots | a_m \quad \text{and}$$
$$\{a_1 y_1, a_2 y_2, \ldots, a_m y_m\} \text{ is a basis of } N.$$

<u>Proof</u> Read Thm 12.1.4 in book, somewhat tedious but constructive.

<u>Example</u> $R = \mathbb{Z}$ $M = \mathbb{Z} \oplus \mathbb{Z}$ $N = 2\mathbb{Z} \oplus 3\mathbb{Z}$

<u>Recall</u> M cyclic $\Rightarrow M \cong Rm \Rightarrow M \cong R/\mathrm{ann}\, m \overset{\text{PID}}{\Longrightarrow} M \cong R/(a)$.

<u>Fundamental Theorem</u> R a PID, M a f.g. R-module.

1. M is $\cong$ to a direct sum of cyclic modules, i.e.

$$M \cong R^r \oplus R/(a_1) \oplus R/(a_2) \oplus \cdots \oplus R/(a_m)$$

where $a_1 | a_2 | \cdots | a_m$ are nonzero, nonunits

2. Thus, M is torsion free $\longleftrightarrow$ M is free

3. $\mathrm{Tor}(M) \cong R/(a_1) \oplus \cdots \oplus R/(a_m)$

M is torsion $\longleftrightarrow$ $r=0$ in which case $\mathrm{Ann}\, M = (a_m)$

<u>Rmks/Defs</u>

1. r is called the <u>Free Rank</u> or <u>Betti #</u>

2. The elements $a_1, a_2, \ldots, a_m$ are unique up to units and are called <u>invariant factors</u> of M.

3. Recall that PID $\to$ UFD. Thus we can factor each a_i into primes and use the chinese rem. thm

$$M \cong R^r \oplus R/(p_1^{\alpha_1}) \oplus R/(p_2^{\alpha_2}) \oplus \cdots \oplus R/(p_s^{\alpha_s})$$

For primes p_i not necessarily distinct.

$p_1^{\alpha_1}, p_2^{\alpha_2}, \ldots, p_s^{\alpha_s}$ are called <u>elementary divisors</u> and are unique.

Sketch of Proof

Choose min set of generators $\{x_1, x_2, \ldots, x_n\}$ of M.
Let $R^n = \langle b_1 \rangle \oplus \cdots \oplus \langle b_n \rangle$ free.
Define $\pi : R^n \twoheadrightarrow M$ by $\pi(b_i) = x_i$, so $M \cong R^n / \ker \pi$.

Apply Thm to submodule $\ker \pi$, $\exists$ basis $y_1, \ldots, y_n$ of R^n such that $\{a_1 y_1, a_2 y_2, \ldots, a_m y_m\}$ is a basis of $\ker \pi$.

$$M \cong Ry_1 \oplus \cdots \oplus Ry_n \Big/ Ra_1y_1 \oplus Ra_2y_2 \oplus \cdots \oplus Ra_my_m \oplus 0 \oplus \cdots \oplus 0$$

$$\cong R/(a_1) \oplus R/(a_2) \oplus \cdots \oplus R/(a_m) \oplus R^{n-m} \quad //$$

Special Case $R = \mathbb{Z}$

Fundamental Theorem of f.g. Abelian Groups

1. Any such A is $\cong \mathbb{Z}^r \oplus \mathbb{Z}/n_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/n_s\mathbb{Z}$ where

   $n_i \mid n_{i+1}$ in a unique way

2. Alternately, suppose $|G| = n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$
   Then
   1. $G \cong A_1 \times A_2 \times \cdots \times A_k$ w/ $|A_i| = p_i^{\alpha_i}$
   2. $A_i \cong \mathbb{Z}/p_i^{\beta_1} \times \mathbb{Z}/p_i^{\beta_2} \times \cdots \times \mathbb{Z}/p_i^{\beta_s}$

      $\beta_1 + \cdots + \beta_s = \alpha_i$

Then $\{p_i^{\beta_i}\}$ are elementary divisors.

<u>Example</u>

$|G| = 72 = 2^3 \cdot 3^2$

	<u>Elt Div</u>	<u>Inv Factors</u>
1. $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \cong \mathbb{Z}_2 \times \mathbb{Z}_6 \times \mathbb{Z}_6$	2, 2, 2, 3, 3	2, 6, 6
2. $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_9 \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{18}$	2, 2, 2, 9	2, 2, 18
3. $G \cong \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \cong \mathbb{Z}_6 \times \mathbb{Z}_{12}$	2, 4, 3, 3	6, 12
4. $G \cong \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_9 \cong \mathbb{Z}_2 \times \mathbb{Z}_{36}$	2, 4, 9	2, 36
5. $G \cong \mathbb{Z}_8 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \cong \mathbb{Z}_3 \times \mathbb{Z}_{24}$	8, 3, 3	3, 24
6. $G \cong \mathbb{Z}_8 \times \mathbb{Z}_9 \cong \mathbb{Z}_{72}$	8, 9	72