3/20/07

Jordan Canonical Form

p. 499
\# 18, 19, 21, 24, 31,

Review

F.g. module over a PID is $\cong$

$R^r \oplus R/(a_1) \oplus \cdots \oplus R/(a_m)$ where $a_1 | a_2 | \cdots | a_m$ invariant factors

Also $R^r \oplus R/(p_1^{\alpha_1}) \oplus \cdots \oplus R/(p_k^{\alpha_k})$

Last Time

$R = F[x]$, module is V acted on by linear map $T: V \to V$
$x \cdot v = T(v)$.

Invariant Factors are polynomials.

- min poly $m_T(x) = a_m(x)$
- char poly $= \prod a_i(x)$
- $\dim V = \sum_i \deg a_i(x)$

- Matrix w.r.t. obvious basis $\{1, x, x^2, \ldots, x^{k-1}\}$ is rational canonical form

Recall To get elementary divisors first factor each $a_i(x)$ into irreducibles. Then "pull off" primes, etc.

i.e.

elt divisors = "maximal" prime power factors of $a_i(x)$.

Special Case

Assume: F is algebraically closed
Thus every irreducible polynomial is degree 1.

Cor Elementary divisors are all of the form $(x-\lambda)^k$.

Cor $V \cong F[x]/(x-\lambda_1)^{k_1} \oplus \cdots \oplus F[x]/(x-\lambda_s)^{k_s}$

Let's study

$F[x]/(x-\lambda)^k$ w/ action of x by mult.

Basis of $F[x]/(x-\lambda)^k$

$$(\bar{x}-\lambda)^{k-1}, (\bar{x}-\lambda)^{k-2}, \ldots, (\bar{x}-\lambda), 1$$

Rmk This is obviously a basis

$$\begin{array}{l} \bar{x}^{k-1} + \cdots \\ \quad \bar{x}^{k-2} + \cdots \\ \qquad \ddots \\ \qquad\quad \bar{x}-\lambda \\ \qquad\qquad 1 \end{array}$$

Problem Matrix with respect to this basis

$$x \cdot (x-\lambda)^{k-1} = (x-\lambda+\lambda)(x-\lambda)^{k-1} = (x-\lambda)^k + \lambda(x-\lambda)^{k-1} \equiv \lambda(x-\lambda)^{k-1}$$

$$x \cdot (x-\lambda)^{k-2} = (x-\lambda)^{k-1} + \lambda(x-\lambda)^{k-2}$$

etc

Lemma The module $V \cong F[x]/(x-\lambda)^k$, in this basis,

has matrix
$$\begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ & \lambda & 1 & \cdots & 0 \\ & & \ddots & \ddots & \\ & & & & 1 \\ & & & & \lambda \end{pmatrix}$$

Def. This $k \times k$ matrix w/ λ's on diag, 1's on super diag is a $k \times k$ <u>elementary Jordan matrix</u> w/ e-value λ.
$J_k(\lambda)$

Lemma $\vdash$ $C.P.(J_k(\lambda)) = M.P.(J_k(\lambda)) = (x-\lambda)^k$

Def A matrix is in <u>Jordan Canonical Form</u> if it is block diagonal of form

$$\begin{pmatrix} J_{k_1}(\lambda_1) & & & 0 \\ & J_{k_2}(\lambda_2) & & \\ & 0 & \ddots & \\ & & & J_{k_s}(\lambda_s) \end{pmatrix}.$$

Theorem Let V be a f.d. v.s. /F, $T: V \to V$ linear. Assume all e-values of T are in F (e.g. $F = \bar{F}$). Then $\exists$ a basis of V for which the matrix of T is in JCF. The JCF is unique up to permuting the blocks.

In particular if $A \in M_{n \times n}(F)$ then $\exists$ $P \in GL_n(F)$ such that PAP^{-1} is in JCF.

Recall HW: $A = \begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_n \end{pmatrix}$

Then $mp(A) = lcm(mp(A_i))$!

Examples $A_1 = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$ $A_2 = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ $A_3 = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}$ $A_4 = (3)$

$$A = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix}$$

$cp(A) = (x-2)^5 (x-3)^4$

$mp(A) = (x-2)^3 (x-3)^3$

Lemma The JCF of a diagonal matrix is itself.

Def A matrix A is diagonalizable (over F) if it is similar to a diagonal matrix via $Gl_n(F)$

Thm Suppose all e-values of A are in F. Then A is diagonalizable iff minimal poly has no repeated roots.

Proof Put in JCF

Algorithm:

1. RCF $\longleftrightarrow$ JCF

Follow rules for elt divisors, inv facts

Example

Suppose $c.p.(A) = (x^4-1)(x^2-1)$

$$= (x-1)^2(x+1)^2(x^2+1)$$

RCFs

Class 1 $\quad (x-1)(x+1), \quad (x-1)(x+1)(x^2+1)$

Class 2 $\quad (x+1), \quad (x+1)(x-1)^2(x^2+1)$

Class 3 $\quad (x-1), \quad (x+1)^2(x-1)(x^2+1),$

Class 4 $\quad (x-1)^2(x+1)^2(x^2+1)$

#1 $\quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\qquad$ #1 $\quad \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & \\ 0 & 1 & \end{pmatrix}$ $\qquad$ #1 $\quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$