

3/27/07

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Algebraic Extensions

Recall

Prop Given K/F and $\alpha \in K$, then α is algebraic \iff $[F(\alpha):F] < \infty$. In this case α has a minimal polynomial of degree $n = [F(\alpha):F]$.

Cor Finite extensions are algebraic.

Thm $F \subset K \subset L$ then $[L:F] = [L:K][K:F]$

Example $\mathbb{Q} \subset \mathbb{Q}(\sqrt{2}) \subset \mathbb{Q}(\sqrt[4]{2})$

Theorem K/F is finite $\iff K = F(\alpha_1, \alpha_2, \dots, \alpha_r)$ with α_i algebraic.
If so the degree is $\leq \deg \alpha_1 + \deg \alpha_2 + \dots + \deg \alpha_r$

Proof \rightarrow Pick α basis
 \leftarrow Get α basis spanning set

Cor α, β alg $\iff \alpha \pm \beta, \alpha\beta, \alpha/\beta$ also are algebraic.

Cor Let L/F be arbitrary. Then the set of elements of L algebraic \iff form a subfield.

Examples

1. Let $\bar{\mathbb{Q}} =$ elts of \mathbb{C} algebraic / \mathbb{Q} . Since

$$[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2 \text{ then this}$$

is an infinite algebraic extension of \mathbb{Q} .

a. Prop: Suppose $K/F, K/F$ are algebraic. Then so is K/F .

Proof Let $\alpha \in L$. So $a_n \alpha^n + a_{n-1} \alpha^{n-1} + \dots + a_1 \alpha + a_0 = 0$ & $a_i \in K$.

Each a_i is algebraic / F so $F(a_0, a_1, \dots, a_n) / F$ is finite.

But $F(a_0, a_1, \dots, a_n) : F(a_0, a_1, \dots, a_n) \leq n$.

Thus $[F(\alpha, a_0, a_1, \dots, a_n) : F] \leq n$ algebraic so α is

Splitting Fields

Def Let $f(x) \in F[x]$. Then K/F is a splitting field for $f(x)$ if $f(x)$ factors into linear terms in $K[x]$ but does not factor completely over any proper subfield of K .

Theorem Let $f(x) \in F[x]$. A splitting field for $f(x)$ exists.

Proof First show \exists a field extension E/F where $f(x)$ splits, by induction on degree.

Then choose $K = \bigcap_{E \text{ w/ } f \text{ splits}} E$ all subfields

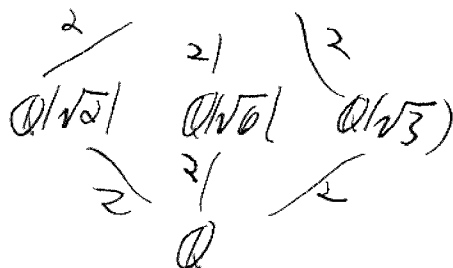
Remark This does not (yet!) show uniqueness.

Def Suppose K/F is algebraic extension which is a splitting field of a collection of polynomials. Then K/F is normal extension.

Examples

1. $x^2 - 2$ over \mathbb{Q} , SF is $\mathbb{Q}(\sqrt{2})$

2. $p(x) = (x^2 - 2)(x^2 - 3)$ SF is $\mathbb{Q}(\sqrt{2}, \sqrt{3})$



3. $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$ is not a normal extension!

$$x^3 - 2 \text{ roots } \sqrt[3]{2}, \sqrt[3]{2} \left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right), \sqrt[3]{2} \left(-\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)$$

Thus S.F. of $x^3 - 2$ is $\mathbb{Q}(\sqrt[3]{2}, \sqrt{-3})$
Degree 6 over \mathbb{Q} .

Thm Degree of S.F. of poly of degree n
is at most $n!$

Rank May of course be smaller.

4. Splitting Field of $x^n - 1$

(Roots are $e^{2\pi i k/n}$ $k=0, 1, \dots, n-1$)
S.F. = $\mathbb{Q}(e^{2\pi i/n})$ called cyclotomic field of n^{th} roots of unity
 $\mathbb{Q}(\zeta_n)$

Problem Degree of this? A: $\varphi(n)$

Special case

$$[\mathbb{Q}(\zeta_p) : \mathbb{Q}] = p-1$$

Thm Splitting Field is unique up to \cong .

SE

Def A field \bar{F} is an algebraic closure of F if \bar{F}/F is algebraic and every $f(x) \in F[x]$ splits in \bar{F} .

Def K is algebraically closed if every polynomial in $K[x]$ has a root in K .

Prop \bar{F} is algebraically closed, i.e. $\overline{\bar{F}} = \bar{F}$.

Proof Let $f(x) \in \bar{F}[x]$ and α a root.

so $\bar{F}(x) = \bar{F}$ algebraic

so $\bar{F}(x) = \bar{F}$ alg.

so $\alpha \in \bar{F}$ //

Thm Algebraic Closures Exist.