3/27/07 $\qquad$ pp. 345 1-5

Algebraic Extensions

Recall

Prop Given K/F and $\alpha \in K$, then α is algebraic $/F$ iff $[F(\alpha):F] < \infty$. In this case α has a minimal polynomial of degree $n = [F(\alpha):F]$

Cor Finite extensions are algebraic.

Thm $F \subset K \subset L$ then $[L:F] = [L:K][K:F]$

Example $\mathbb{Q} \subset \mathbb{Q}(\sqrt{2}) \subset \mathbb{Q}(\sqrt[4]{2})$

Theorem K/F is finite $\longleftrightarrow K = F(\alpha_1, \alpha_2, \ldots, \alpha_k)$ with α_i algebraic. If so the degree is $\leq \deg\alpha_1 \deg\alpha_2 \cdots \deg\alpha_k$

Proof $\rightarrow$ Pick a basis

$\leftarrow$ Get a basis spanning set

Cor α, β alg $/F \Rightarrow \alpha \pm \beta, \alpha\beta, \alpha/\beta$ also are algebraic.

Cor Let L/F be arbitrary. Then the set of elements of L algebraic $/F$ form a subfield.

<u>Examples</u>

1. Let $\overline{\mathbb{Q}}$ = elts of $\mathbb{C}$ algebraic$/\mathbb{Q}$. Since

$$[\mathbb{Q}(\sqrt[n]{2}):\mathbb{Q}]=n$$ then this

is an infinite algebraic extension of $\mathbb{Q}$.

2. Prop: Suppose L/K, K/F are algebraic. Then so is L/F.

<u>Proof</u> Let $\alpha \in L$. So $a_n\alpha^n + a_{n-1}\alpha^{n-1} + \dots + a_1\alpha + a_0 = 0$ $a_i \in K$.

Each a_i is algebraic $/F$ so $F(a_0, a_1, \dots, a_n)/F$ is finite.

But $F(\alpha, a_0, a_1, a_2, \dots, a_n) : F(a_0, a_1, \dots, a_n) \le n$.

Thus $[F(\alpha, a_0, a_1, \dots, a_n) : F]$ is algebraic so α is.

Splitting Fields

Def Let $f(x) \in F[x]$. Then K/F is a splitting field for $f(x)$ if $f(x)$ factors into linear terms in $K[x]$ but does not factor completely over any proper subfield of K.

Theorem Let $f(x) \in F[x]$. A splitting field for $f(x)$ exists.

Proof First show $\exists$ a field extension E/F where $f(x)$ splits, by induction on degree.
Then choose $K = \bigcap$ all subfields E w/ F and all roots.

Rmk This does not (yet!) show uniqueness.

Def Suppose K/F is algebraic extension which is a splitting field of a collection of polynomials. Then K/F is normal extension.

Examples

1. x^2-2 over $\mathbb{Q}$, S.F. is $\mathbb{Q}(\sqrt{2})$

2. $p(x) = (x^2-2)(x^2-3)$ S.F. is $\mathbb{Q}(\sqrt{2}, \sqrt{3})$

$$\begin{tikzcd}
 & \mathbb{Q}(\sqrt{2},\sqrt{3}) \arrow[dl, dash, "2"'] \arrow[d, dash, "2"] \arrow[dr, dash, "2"] & \\
\mathbb{Q}(\sqrt{2}) \arrow[dr, dash, "2"'] & \mathbb{Q}(\sqrt{6}) \arrow[d, dash, "2"] & \mathbb{Q}(\sqrt{3}) \arrow[dl, dash, "2"] \\
 & \mathbb{Q} &
\end{tikzcd}$$

3. $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$ is not a normal extension!

x^3-2 roots $\sqrt[3]{2}$, $\sqrt[3]{2}\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)$, $\sqrt[3]{2}\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)$

Thus S.F. of x^3-2 is $\mathbb{Q}(\sqrt[3]{2}, \sqrt{-3})$

Degree 6 over $\mathbb{Q}$

<u>Thm</u> Degree of S.F. of poly of degree n is at most $n!$

<u>Rmk</u> May of course be smaller.

4. <u>Splitting Field of x^n-1</u>

(Roots are $e^{2\pi i k/n}$ $k = 0, 1, \ldots, n-1$

(S.F. $= \mathbb{Q}(e^{2\pi i/n}) = \mathbb{Q}(\zeta_n)$ called cyclotomic field of n^{th} roots of unity

<u>Problem</u> Degree of this? A: $\varphi(n)$

<u>Special Case</u>

$$[\mathbb{Q}(\zeta_p):\mathbb{Q}] = p-1$$

<u>Thm</u> Splitting Field is unique up to $\cong$.

Def A field $\overline{F}$ is an algebraic closure of F if $\overline{F}/F$ is algebraic and every $f(x) \in F[x]$ splits in $\overline{F}$.

Def K is algebraically closed if every polynomial in $K[x]$ has a root in K

Prop $\overline{F}$ is algebraically closed, i.e. $\overline{\overline{F}} = \overline{F}$.

Proof Let $f(x) \in \overline{F}[x]$ α a root.

so $\overline{F}(\alpha) : \overline{F}$ algebraic

so $\overline{F}(\alpha) : F$ alg.

so $\alpha \in \overline{F}$. //

Thm Algebraic Closures Exist.