Math 8310 Spring 2007 Midterm Exam #1.

Part I: Do all five problems.

1. Let p be prime. Prove that $\frac{x^p-1}{x-1}$ is irreducible in $\mathbb{Z}[x]$.

2. For which integers n is x^3+nx+2 irreducible in $\mathbb{Z}[x]$.

3. Let R be a ring and G an infinite multiplicative cyclic group with generator x. Is the group ring $R(G)$ isomorphic to the polynomial ring in one indeterminate over R? Explain.

4. Give an example of a ring R and a finitely generated R module that is not finitely generated as an abelian group.

5. State three equivalent conditions for an R module P to be *projective.*

Part II: Choose 3 out of 5.

7. Suppose $f: M \to N$ is an R-module homomorphism. Prove that f is one-to one if and only for every R-module D and every pair of R-module homomorphisms $g, h: D \to M$ such that $fg = fh$, we have $g = h$.

8. Let $0 \to A \to B \xrightarrow{f} C \to 0$ and $0 \to C \xrightarrow{g} D \to E \to 0$ be short exact sequences of R-modules.

a. Show that $0 \to A \to B \xrightarrow{gf} D \to E \to 0$ is exact.

b. Show that every exact sequence may be obtained by splicing together suitable short exact sequences in this manner.

9. Suppose $f: M \to N$ and $g: N \to M$ are R-module homomorphisms such that $gf = 1_M$. Prove that $N = \operatorname{Im} f \oplus \operatorname{Ker} g$.

10. Suppose I is a right ideal of a ring R with identity and B is a left R-module. Prove there is a group isomorphism:

$$R/I \otimes_R B \cong B/IB$$

where IB is the subgroup of B generated by all elements of the form $\{ib \mid i \in I\}$.

11. Let M be a right R module and N a left R module. Describe in detail the construction of the tensor product $M \otimes_R N$. What structure does it have?

8310 Midterm 1 Solutions

1. Let $f(x) = \frac{x^p - 1}{x-1}$ Then $f(x+1) = \frac{(x+1)^p - 1}{x} = x^{p-1} + px^{p-2} + \binom{p}{2}x^{p-3} + \dots + p$ is irreducible by Eisenstein criterion. Thus $f(x)$ is as well since any factorization of $f(x) = g(x)h(x)$ yields $f(x+1) = g(x+1)h(x+1)$.

2. $x^3 + nx + 2$ factors / $\mathbb{Z}[x]$ iff it has a root in $\mathbb{Z}$ and the root must be $\pm 2, \pm 1$.

   $f(1) = 3+n \quad f(2) = 10+2n \quad f(-1) = 1-n \quad f(-2) = -6-2n.$

   Thus it is irreducible iff $n \neq -3, -5, 1$.

3. No, the units in polynomial ring $R[x]$ are units in R. However the group of units in $R(G)$ includes all these plus many more, e.g. X.

4. $R = M = \mathbb{Q}$ is cyclic as a $\mathbb{Q}$-module, not f.g. as an abelian group.

5. P is projective is equivalent to:

   a. $Hom_R(P, -)$ is exact.

   b. P is a direct summand of a free module.

   c. Any SES $0 \to M \to N \to P \to 0$ splits

   d. Given
   $$\begin{array}{ccccc} & & P & & \\ & & \downarrow f & & \\ M & \xrightarrow{\varphi} & N & \to & 0 \end{array}$$
   $\exists\, \tilde{f}: P \to M$ such that $\varphi \circ \tilde{f} = f$.

7. $\rightarrow$ Suppose f is 1-1. Let $d \in D$. $f(g(d)) = f(h(d))$ by assumption so $g(d) = h(d)$ since f is 1-1 so $g = h$ since d was arbitrary.

$\leftarrow$ Let $i: \text{Ker} f \hookrightarrow M$ be inclusion map, so i is 1-1.
$z: \text{Ker} f \rightarrow M$ be the zero map.

Then $f \circ i = f \circ z \overset{=0}{}$ ~~by assumption~~, as so $i = z$ by assumption. Hence $\text{Ker} f = 0$ so f is 1-1

8. $0 \rightarrow A \xrightarrow{\psi} B \xrightarrow{f} C \rightarrow 0 \qquad 0 \rightarrow C \xrightarrow{g} D \xrightarrow{\varphi} E \rightarrow 0$

a. Show $0 \rightarrow A \xrightarrow{\psi} B \xrightarrow{g \circ f} D \xrightarrow{\varphi} E \rightarrow 0$ is exact.

Proof ψ is 1-1 and φ is onto. Thus we must show $\text{Ker}\, g \circ f = \text{Im}\, \psi$ and $\text{Im}\, g \circ f = \text{Ker}\, \varphi$

Since g is 1-1, $\text{Ker}(g \circ f) = \text{Ker} f = \text{Im}\, \psi$ by exactness of 1st SES
Since f is onto, $\text{Im}\, g \circ f = \text{Im}\, g = \text{Ker}\, \varphi$ by exactness of 2nd SES
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b. Now suppose we have an exact sequence

$A_{i-1} \xrightarrow{f} A_i \xrightarrow{g} A_{i+1} \xrightarrow{h} A_{i+2}$

Notice that we have SES $0 \rightarrow \text{Im} f \xrightarrow{i} A_i \xrightarrow{g} \text{Im}\, g \rightarrow 0$ since $\text{Ker}\, g = \text{Im} f$. Also we have

$0 \rightarrow \text{Ker}\, h \xrightarrow{i} A_{i+1} \xrightarrow{h} \text{Im}\, h \rightarrow 0$. But $\text{Im}\, g = \text{Ker}\, h$ so

splicing gives ~~$0 \rightarrow \text{Ker}\, h \xrightarrow{i}$~~ A_{i+1}

$0 \rightarrow \text{Im} f \xrightarrow{i} A_i \xrightarrow{g} A_{i+1} \xrightarrow{h} \text{Im}\, h \rightarrow 0$. Now keep splicing!

9 $\operatorname{Im} f$, $\operatorname{Ker} g$ are submodules. Let $f(m) \in \operatorname{Im} f \cap \operatorname{Ker} g$. Then $0 = g(f(m)) = m$ so $f(m) = 0$ so $\operatorname{Im} f \cap \operatorname{Ker} g = 0$ $(*)$

Let $n \in N$. Then $n = f(g(n)) + n - f(g(n))$

and $g(n - f(g(n))) = g(n) - gfg(n) = g(n) - g(n) = 0$

Thus $n \in \operatorname{Im} f + \operatorname{Ker} g$ so $N = \operatorname{Im} f + \operatorname{Ker} g$ $(**)$.

$(*)$ and $(**) \longrightarrow N = \operatorname{Im} f \oplus \operatorname{Ker} g$.

10. Define $\psi : R/I \times B \to B/IB$ by $\psi(\bar{r}, b) = \overline{rb}$ where $\bar{r} = r + I$.

First observe that $\bar{r_1} = \bar{r_2} \leftrightarrow r_1 - r_2 \in I \Rightarrow (r_1 - r_2)b \in IB$
$\to r_1 b + IB = r_2 b + IB$

so this map is well-defined. One easily checks it is balanced, for example $\psi(\bar{r} r_1, b) = \overline{r r_1 b} = \psi(\bar{r}, r_1 b)$.

Thus we get a group homomorphism $\tilde{\psi} : R/I \otimes B \to B/IB$ defined by $\tilde{\psi}(\bar{r} \otimes b) = \overline{rb}$.

Consider the map $\varphi : B/IB \to R/I \otimes B$ given by $\varphi(\bar{b}) = 1 \otimes b$. Again notice this is well defined, for suppose $\bar{b_1} = \bar{b_2}$, so $b_1 - b_2 = ib$ so $1 \otimes (b_1 - b_2) = 1 \otimes ib = i \otimes b = 0$.

$$1 \otimes (b_1 - b_2) = 1 \otimes b_1 - 1 \otimes b_2$$

φ is clearly an abelian group homomorphism.

Finally note that $\varphi \circ \tilde{\psi}(\bar{r} \otimes b) = \varphi(\overline{rb}) = 1 \otimes rb = r \otimes b$

And $\tilde{\psi} \circ \varphi(\bar{b}) = \tilde{\psi}(1 \otimes \bar{b}) = \bar{b}$ so φ & $\tilde{\psi}$ are $\cong$'s

11. Let F be the free abelian group on the set $M \times N$.

Let K be the subgroup generated by
$$\left\{ \begin{array}{l} (m_1+m_2, n) - (m_1, n) - (m_2, n), \\ (m, n_1+n_2) - (m, n_1) - (m, n_2), \\ (m, rn) - (mr, n) \end{array} \right\}$$

Then

$$M \otimes_R N := F/K.$$ In general it has only the structure of an abelian group.