Math 8310 Spring 2007 Midterm Exam #2.

Each part is worth 1/3 of the final score.

Part I: "Short" answer. Do all five problems.

1. Let V be vector space. Show that there is a natural injective linear transformation from V to V^{**}, which is an isomorphism if V is finite-dimensional.

2. Which of the following matrices are in rational canonical form? Justify your answer.

$$A := \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 6 \\ 0 & 1 & -1 \end{pmatrix} \quad B := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 6 \\ 0 & 1 & -1 \end{pmatrix} \quad C := \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

3. Let V be a 4-dimensional vector space with basis $\{v_1, v_2, v_3, v_4\}$. Give a basis for $\mathrm{Sym}^2(V)$ and $\Lambda^2(V)$.

4. Show that any finite field extension is algebraic.

5. Write down a 5×5 matrix which has minimal polynomial $(x-2)^2(x-1)$ and characteristic polynomial$(x-2)^3(x-1)^2$.

Part II: Do this problem.

6. State the fundamental theorem for finitely generated modules over a PID. Discuss the invariant factor form and elementary divisor form. Explain how this theorem can be used to prove the existence of rational canonical and Jordan canonical form for a matrix $A \in M_n(K)$. Include a discussion of what, if any, requirements on the field K are necessary for each.

Part III: Choose 2 out of 4.

7. Find all possible rational canonical forms for a 6×6 matrix in $M_6(\mathbb{Q})$ which has minimal polynomial $(x-2)^2(x+3)$.

8. Suppose K is not a perfect field. Prove there exists irreducible inseparable polynomials over K. Conclude that there exist inseparable finite extensions of K.

9. Let V be an n-dimensional vector space over K and let $\psi : V \to V$ be a linear map. Describe the determinant of ψ in terms of certain exterior powers of V.

10. Let K be an extension of F with $[K : F] = n$. Prove that K is isomorphic to a subfield of the ring of $n \times n$ matrices over F.

MIDTERM #2 SOLUTIONS

1. Define $E: V \to V^{**}$ by $E_v(f) = f(v)$ for $f \in V^*$. Clearly E is a linear map. Suppose $v \neq 0$ and extend v to a basis of V. Define $f \in V^*$ by $f(v) = 1$ and $f = 0$ on the rest of the basis. Then $E_v f \neq 0$, so E_v is injective. When V is f.d. Then $\dim V = \dim V^* = \dim V^{**}$ so E_V is an $\cong$.

2. A is, inv. factors $(x-2)$, x^2+x-6
B is not since $x-1 \nmid x^2+x-6$
C is not, it is in JCF.

3. Recall $Sym^2(V)$ and $\Lambda^2(V)$ are both quotients of $V \otimes V$.

$$Sym^2(V) = \langle v_1 \otimes v_1, v_2 \otimes v_2, v_3 \otimes v_3, v_4 \otimes v_4, v_1 \otimes v_2, v_1 \otimes v_3, v_1 \otimes v_4, v_2 \otimes v_3, v_2 \otimes v_4, v_3 \otimes v_4 \rangle$$

$$\Lambda^2(V) = \langle v_1 \wedge v_2, v_1 \wedge v_3, v_1 \wedge v_4, v_2 \wedge v_3, v_2 \wedge v_4, v_3 \wedge v_4 \rangle$$

4. Suppose $[K:F] = n$. Let $\alpha \in K$. Then $1, \alpha, \alpha^2, \ldots, \alpha^n$ must be linearly dependent $/F$, so α satisfies a polynomial w/coefs in F. Thus K/F is algebraic.

5. $$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

C.P. $(x-2)^3(x-1)^2$
M.P. $(x-2)^2(x-1)$

6. <u>Thm</u> Let R be a PID, M a f.g. R-module. Then

$$M \cong R^m \oplus R/(a_1) \oplus \cdots \oplus R/(a_k) \quad \text{where } a_1 | a_2 | \cdots | a_k$$

are nonzero nonunits and are unique up to units. They are the <u>invariant factors</u>. Factoring the $\{a_i\}$ into irreducibles and applying the Chinese Remainder Thm gives the elementary divisor form:

$$M \cong R^m \oplus R/(p_1^{\alpha_1}) \oplus \cdots \oplus R/(p_s^{\alpha_s}) \quad \text{where } p_i \text{ are}$$

not necessarily distinct primes.

Now suppose F is a field and V is an $F[x]$ module. One easily shows V is a vector space on which x acts as a linear transformation. Putting V in invariant factor form and choosing the basis $\{\bar{1}, \bar{x}, \ldots, \bar{x}^e\}$ for $F[x]/(a_i(x))$ gives us the matrix in RCF.

If all the e-values of the char poly are in F then each invariant factor factors completely into linear terms. Thus $p_i(x) = (x-\lambda_i)$ so

$$V \cong F[x]/(x-\lambda_1)^{k_1} \oplus \cdots \oplus F[x]/(x-\lambda_e)^{k_e}$$

Choosing the basis $\{1, \overline{(x-\lambda)}, \ldots, \overline{(x-\lambda)}^{k-1}\}$ for each summand, we get the matrix in JCF.

7. Let's determine possible inv. factors

Class 1: $(x-2)^2(x+3),\ (x-2)^2(x+3) = x^3-x^2-8x+12,\ x^3-x^2-8x+12$

2: $(x-2),\ (x-2)^2,\ (x-2)^2(x+3) = x-2,\ x^2-4x+4,\ x^3-x^2-8x+12$

3: $(x-2),\ (x-2)(x+3),\ (x-2)^2(x+3) = x-2,\ x^2+x-6,\ x^3-x^2-8x+12$

4: $(x+3),\ (x-2)(x+3),\ (x-2)^2(x+3) = x+3,\ x^2+x-6,\ x^3-x^2-8x+12$

5: $(x-2),\ (x-2),\ (x-2),\ (x-2)^2(x+3)$

6: $(x+3),\ (x+3),\ (x+3),\ (x-2)^2(x+3)$

1. $\begin{pmatrix} 0 & 0 & -12 & & & \\ 1 & 0 & 8 & & & \\ 0 & 1 & 1 & & & \\ & & & 0 & 0 & -12 \\ & & & 1 & 0 & 8 \\ & & & 0 & 1 & 1 \end{pmatrix}$
2. $\begin{pmatrix} 2 & & & & & \\ & 0 & -4 & & & \\ & 1 & 4 & & & \\ & & & 0 & 0 & -12 \\ & & & 1 & 0 & 8 \\ & & & 0 & 1 & 1 \end{pmatrix}$
3. $\begin{pmatrix} 2 & & & & & \\ & 0 & 6 & & & \\ & 1 & -1 & & & \\ & & & 0 & 0 & -12 \\ & & & 1 & 0 & 8 \\ & & & 0 & 1 & 1 \end{pmatrix}$

4. $\begin{pmatrix} -3 & & & & & \\ & 0 & 6 & & & \\ & 1 & -1 & & & \\ & & & 0 & 0 & -12 \\ & & & 1 & 0 & 8 \\ & & & 0 & 1 & 1 \end{pmatrix}$
5. $\begin{pmatrix} 2 & & & & & \\ & 2 & & & & \\ & & 2 & & & \\ & & & 0 & 0 & -12 \\ & & & 1 & 0 & 8 \\ & & & 0 & 1 & 1 \end{pmatrix}$
6. $\begin{pmatrix} -3 & & & & & \\ & -3 & & & & \\ & & -3 & & & \\ & & & 0 & 0 & -12 \\ & & & 1 & 0 & 8 \\ & & & 0 & 1 & 1 \end{pmatrix}$

8. Suppose K is not perfect. Choose $a \in K$, $a \notin K^p$. Then $x^p - a$ has no roots in K. Choose α a root in an extension field, so $\alpha^p = a$. Thus $(x-\alpha)^p = x^p - \alpha^p = x^p - a$, so $p(x) = x^p - a$ is inseparable. Suppose $p(x)$ is reducible in $K[x]$. Then

$$q(x) = (x-\alpha)^s \in K[x] \text{ for some } 1 \le s < p$$

But

$$q(x) = x^s - s\alpha x^{s-1} \dots \text{ and } s\alpha \neq 0 \text{ since } 1 \le s < p.$$

Thus $s\alpha \in K \Rightarrow \alpha \in K$ a contradiction

Hence $x^p - a$ is an irreducible, inseparable polynomial!

9. $\psi : V \to V$ induces a linear map $\Lambda^k \psi : \Lambda^k V \to \Lambda^k V$. But $\Lambda^n V$ is 1-dimensional so $\Lambda^n \psi$ is just multiplication by some scalar.

This scalar is $\det \psi$.

10. Let $[K:F]=n$. Thus $K \cong F^n$ as F vector spaces. For $\alpha \in K$ let $L_\alpha : K \to K$ be $L_\alpha(a) = \alpha a$. This is a linear map $F^n \to F^n$ which is ~~non~~ invertible for $\alpha \neq 0$. Clearly $L_{\alpha\beta} = L_\alpha L_\beta$, $L_{\alpha+\beta} = L_\alpha + L_\beta$

Thus, choosing a basis of K/F, L_α induces

a field map $K \hookrightarrow M_n(F)$

Thus K is $\cong$ to a subfield of the ring $M_n(F)$.