8-5-11 Solutions Name ____________________

There are 200 possible points. A non graphing calculator and a formula sheet are allowed. Check that there are 8 (2 sided) pages.

1. Find an equation for the tangent plane to the surface

$$x \cos z + y^2 e^{xz} = 4 \quad \text{at} \quad P_0(3, -1, 0)$$

(12)

The given surface is a level surface of the function $F(x, y, z) = x \cos z + y^2 e^{xz}$. Therefore the normal to the surface is given by

$$\nabla F = (\cos z + y^2 z e^{xz})\vec{i} + 2ye^{xz}\vec{j} + (-x \sin z + xy^2 e^{xz})\vec{k}$$

so that $\nabla F(3, -1, 0) = \vec{i} - 2\vec{j} + 3\vec{k}$. An equation of the tangent plane is therefore $(x - 3) - 2(y + 1) + 3z = 0$ or $x - 2y + 3z = 5$.

(15) 2. Solve the initial value problem for $\vec{r}$ as a vector function of t.

$$\begin{aligned} \frac{d\vec{r}}{dt} &= \frac{1}{(t+1)^2}\vec{i} + \frac{1}{t+1}\vec{j} + (t+1)^{1/2}\vec{k} \\ r(\vec{0}) &= \vec{j} + 2\vec{k} \end{aligned}$$

We can find $\vec{r}$ by integrating in t.

$$\vec{r} = \int \frac{1}{(t+1)^2}\vec{i} + \frac{1}{t+1}\vec{j} + (t+1)^{1/2}\vec{k}\, dt = -\frac{1}{t+1}\vec{i} + \ln|t+1|\vec{j} + \frac{2}{3}(t+1)^{3/2}\vec{k} + \vec{C}$$

and we set $t = 0$ to find $\vec{C}$: $\vec{j} + 2\vec{k} = \vec{r}(0) = -\vec{i} + \frac{2}{3}\vec{k} + \vec{C}$. Therefore $\vec{C} = \vec{i} + 2\vec{j} + \frac{4}{3}\vec{k}$ and

$$\vec{r} = -\frac{1}{t+1}\vec{i} + \ln|t+1|\vec{j} + \frac{2}{3}(t+1)^{3/2}\vec{k} + \vec{i} + 2\vec{j} - \frac{2}{3}\vec{k}$$

3. (a) Find the directional derivative of f at (0,1) in the direction of the vector $\vec{v} = \vec{i} + 2\vec{j}$.

(16)

$$f(x, y) = x^2 y + y^2 + ye^{xy}$$

We need $\nabla f = (2xy + y^2 e^{xy})\vec{i} + (x^2 + 2y + e^{xy} + xye^{xy})\vec{j}$ so that $\nabla f(0, 1) = \vec{i} + 3\vec{j}$. The directional derivative is therefore

$$D_{\vec{v}} f(0, 1) = \frac{(\nabla f(0, 1)) \cdot \vec{v}}{|\vec{v}|} = \frac{7}{\sqrt{5}}$$

(b) Find the maximum rate of change of f at (0,1) and the direction in which it occurs if f is the function in part (a).

The maximum rate of change of f is in the direction $\nabla f / |\nabla f|$:

$$\frac{\vec{i} + 3\vec{j}}{\sqrt{10}}$$

and the rate of change in that direction is $\sqrt{10}$.

(18) 4. Find the local maximum, minimum and saddle points for the function $f(x,y) = 6x^2 - 2x^3 + 3y^2 + 6xy$.

Find the critical points. We compute $\nabla f = (12x - 6x^2 + 6y)\vec{i} + (6y + 6x)\vec{j}$. The ∇f is defined everywhere and so the critical points arise only when $\nabla f = \vec{0}$ that is when

$$\begin{aligned} 12x - 6x^2 + 6y &= 0 \\ 6x + 6y &= 0 \end{aligned}$$

From the second equation we see that $y = -x$ and we substitute that into the first equation $6x - 6x^2 = 0$ so that $x = 0$ or $x = 1$. The critical points are (0,0) and (1,-1). We test them for being max, min or saddle points. We compute the second partials: $f_{xx} = 12 - 12x$ $f_{yy} = 6$ and $f_{xy} = 6$ so that the discriminant is $\Delta = f_{xx}f_{yy} - (f_{xy})^2$. At (0,0) $\Delta = 36 > 0$ and $f_{xx} > 0$ and so (0,0) is a local minimum. At (1,-1), $\Delta = -36 < 0$ and so (1,-1) is a saddle.

(17) 5. Find the maximum and minimum values $f(x,y) = xy$ can take on the ellipse $4x^2+y^2 = 16$.

Apply the method of Lagrange Multipliers. The constraint here is $g = 16$ where $g(x,y) = 4x^2 + y^2$. Set $\nabla f = \lambda \nabla g$ for some λ.

$$\begin{aligned} y &= \lambda(8x) \\ x &= \lambda(2y) \\ 4x^2 + y^2 &= 16 \end{aligned}$$

Substituting the second equation into the first gives $y = 16(\lambda)^2 y$ so that either $y = 0$ or $\lambda = \pm 1/4$. Consider first the case $y = 0$. Then $x = 0$ by the first two equations but this does not satisfy the third equation and so $y \neq 0$. Next consider the case $\lambda = 1/4$. The first (and second) equation then says $y = 2x$ so that the third equation says $4x^2 + 4x^2 = 16$ so that $x = \pm\sqrt{2}$ but $y = 2x$ and so we get two possible solutions

$$(\sqrt{2}, 2\sqrt{2}) \text{ and } (-\sqrt{2}, -2\sqrt{2})$$

. The remaining case to consider is $\lambda = -1/4$ in which case we again get $x = \pm\sqrt{2}$ from the third equation but this time the points are

$$(\sqrt{2}, -2\sqrt{2}) \text{ and } (-\sqrt{2}, 2\sqrt{2})$$

To determine the maximum and minimum, we evaluate f: $f(\sqrt{2}, 2\sqrt{2}) = 4 = f(-\sqrt{2}, -2\sqrt{2})$ and $f(\sqrt{2}, -2\sqrt{2}) = -4 = f(-\sqrt{2}, 2\sqrt{2})$ so that the maximum value is 4 and the minimum is -4.

(16) 6. Evaluate the integral $\iiint_D 4y\, dV$ if D is bounded by the elliptic paraboloid $z = 3x^2+y^2+1$ and by the planes $z = 0$, $y = 2x$, $y = 0$ and $x = 1$.

Sketch D. The solid is bounded above by the paraboloid $z = 3x^2 + y^2 + 1$, below by the xy-plane ($z = 0$) and lies above the triangular region with edges $y = 2x$, $y = 0$ (the x-axis) and $x = 1$. Therefore, expanding the triple integral as an iterated integral we have.

$$\begin{aligned}
\iiint_E 4y\,dV &= \int_0^1 \int_0^{2x} \int_0^{3x^2+y^2+1} 4y\,dz\,dy\,dx \\
&= \int_0^1 \int_0^{2x} 4yz|_0^{3x^2+y^2+1}\,dy\,dx \\
&= \int_0^1 \int_0^{2x} 12x^2y + 4y^3 + 4y\,dy\,dx \\
&= \int_0^1 6x^2y^2 + y^4 + 2y^2|_0^{2x}\,dx \\
&= \int_0^1 24x^4 + 16x^4 + 8x^2\,dx = [8x^5 + \frac{8}{3}x^3|_0^1 = \frac{31}{3}
\end{aligned}$$

7. Let D be the solid that lies between the spheres $x^2+y^2+z^2 = 1$ and $x^2+y^2+z^2 = 9$ in the first octant. Express $\iiint_D xz\,dV$ as an iterated (triple) integral in spherical coordinates.
(16) Do NOT evaluate.

Convert to spherical coordinates: the integrand is $xz = (\rho\sin\phi\cos\theta)(\rho\cos\phi)$. Draw a picture of D: it is an eighth of a ball. D can be described as $D = \{(\rho,\theta,\phi) : 1 \le \rho \le 3, 0 \le \phi \le \pi/2, 0 \le \theta \le \pi/2\}$. Therefore

$$\begin{aligned}
\iiint_E xz\,dV &= \int_0^{\pi/2} \int_0^{\pi/2} \int_1^3 \rho\sin\phi\cos\theta)(\rho\cos\phi\rho^2\sin\phi\,d\rho\,d\phi\,d\theta \\
&= \int_0^{\pi/2} \int_0^{\pi/2} \int_1^3 \rho^4(\sin\phi)^2\cos\phi\cos\theta\,d\rho\,d\phi\,d\theta
\end{aligned}$$

8. (a) Show that $\vec{F}(x,y) = (ye^x + \sin y)\vec{i} + (e^x + x\cos y + 2y)\vec{j}$ is conservative and find a
(18) function f so that $\vec{F} = \nabla f$.

We must check if $P_y = Q_x$ where $P = ye^x + \sin y$ and $Q = e^x + x\cos y + 2y$. Since $P_y = e^x + \cos y$ and $Q_x = e^x + \cos y$, we see these are equal (on all of the xy-plane) and so $\vec{F}$ is exact. To find f so that $\vec{F} = \nabla f$ we integrate P with respect to x, regarding y as a constant.

$$f = ye^x + x\sin y + h(y)$$

where $h(y)$ is the constant of integration. We should have $f_y = Q$ and so we differentiate the above expression for f with respect to y and compare to Q: $f_y = e^x + x\cos y + h'(y)$ so that

$$f_y = e^x + x\cos y + h'(y) = Q = e^x + x\cos y + 2y$$

so that $h'(y) = 2y$. Integrating in y, we have $h(y) = y^2 + C$. Substituting into our earlier equation for f we have

$$f = ye^x + x\sin y + y^2 + C$$

(b) Find the work done by $\vec{F}$ ($\vec{F}$ as in part (a)) in moving an object along a curve C from $(0,1)$ to $(2,\pi)$.

The work done is $f(2,\pi) - f(0,1)$ by the vector form of the fundamental theorem of calculus. That is work is $\pi e^2 + 2\sin\pi + \pi^2 + C - (e^0 + 0\sin 1 + 1^2 + C) = \pi e^2 + \pi^2 - 2$.

9. Evaluate the line integral $\int_C \vec{F}\cdot d\vec{r}$ if $\vec{F}(x,y,z) = xyz\vec{i} - xy\vec{j} + x^2\vec{k}$ along the path C
(14) given by $\vec{r}(t) = t\vec{i} + t^2\vec{j} + t^3\vec{k}$, $0 \le t \le 2$.

This is not a closed curve and so Green's theorem does not apply. Also $\vec{F}$ is not exact and so we simply use the straightforward method of calculating line integrals. We need $\vec{r}'(t) = \vec{i} + 2t\vec{j} + 3t^2\vec{k}$ and $\vec{F}(\vec{r}(t)) = tt^2t^3\vec{i} - tt^2\vec{j} + t^2\vec{k} = t^6\vec{i} - t^3\vec{j} + t^2\vec{k}$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^2 (t^6\vec{i} - t^3\vec{j} + t^2\vec{k})\cdot(\vec{i} + 2t\vec{j} + 3t^2\vec{k})\,dt = \int_0^2 t^6 + t^4\,dt = \frac{t^7}{7} + \frac{t^5}{5} = \frac{(32)(27)}{35} = \frac{864}{35}$$

10. Use Green's Theorem to evaluate the line integral

$$\int_C (xe^x + 4x^3y)\,dx + (x^4 + 2xy)\,dy$$

where C is the boundary of the triangle $0 \le x \le 2y$, $0 \le y \le 1$ and is positively oriented.

(16)

Sketch the triangle. Green's theorem says, that if D denotes the triangle $\int_C P\,dx + Q\,dy =$

$\iint_D Q_x - P_y \, dA$. In our case $Q_x - P_y = (4x^3 + 2y) - 4x^3 = 2y$. Therefore

$$\begin{aligned}\int_C (xe^x + 4x^3 y)\, dx + (x^4 + 2xy)\, dy &= \iint_D 2y \, dA = \int_0^1 \int_0^{2y} 2y \, dx \, dy \\ &= \int_0^1 2xy|_0^{2y} \, dy = \int_0^1 4y^2 \, dy = \frac{4}{3} y^3|_0^1 = \frac{4}{3}\end{aligned}$$

(10) 11. Let $\vec{F} = x^2 y\vec{i} + 2y^3 z\vec{j} + 5xz\vec{k}$

(a) Find the curl of $\vec{F}$

The curl is

$$\nabla \times \vec{F} = \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 y & 2y^3 z & 5xz \end{bmatrix} = (0 - 2y^3)\vec{i} - (5z - 0)\vec{j} + (0 - x^2)\vec{k}$$

(b) Find the divergence of $\vec{F}$

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x} M + \frac{\partial}{\partial y} N + \frac{\partial}{\partial z} P = 2xy + 6y^2 z + 5x$$

(16) 12. Find the area of the cap of the sphere $x^2 + y^2 + z^2 = 2$ cut by the cone $z = \sqrt{x^2 + y^2}$

We can treat the surface as the graph of a function $z = \sqrt{2 - x^2 + y^2}$ or describe it parametrically as $\vec{r}(r, \theta) = (r\cos\theta, r\sin\theta, \sqrt{2 - r^2}$ or, alternatively $\vec{r}(\phi, \theta) = (\sqrt{2}\sin\phi\cos\theta, \sqrt{2}\sin\phi\sin\theta, \sqrt{2}$ c
The cone intersects the surface when $x^2 + y^2 + (\sqrt{x^2 + y^2})^2 = 2$ or $x^2 + y^2 = 1$ and that means $z = 1$ (a circle of radius 1 in the plane $z = 1$.) In the first case the area is

$$\iint_R \sqrt{1 + z_x^2 + z_y^2} \, dA = \int_R \frac{\sqrt{2}}{\sqrt{2 - x^2 - y^2}} \, dA = \int_0^{2\pi} \int_0^1 \frac{\sqrt{2}}{\sqrt{2 - r^2}} r \, dr \, d\theta = 2(2 - \sqrt{2})\pi$$

where R is the unit circle in the xy–plane and we have converted to polar coordinates and substituting $u = 2 - r^2$.

Alternatively we can describe the surface it parametrically as $\vec{r}(r, \theta) = (r\cos\theta, r\sin\theta, \sqrt{2 - r^2})$ where $0 \leq r \leq 1$ and $0 \leq \theta \leq 2\pi$ and we can compute the normal

$$\vec{r}_r \times \vec{r}_\theta = \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\theta & \sin\theta & -r(2 - r^2)^{-1/2} \\ -r\sin\theta & r\cos\theta & 0 \end{bmatrix} = \frac{r^2}{(2 - r^2)^{1/2}} \cos\theta\vec{i} + \frac{r^2}{(2 - r^2)^{1/2}} \sin\theta\vec{j} + r\vec{k}$$

The area is therefore

$$\begin{aligned}\int_0^{2\pi}\int_0^1 |\vec{r}_r \times \vec{r}_\theta|\,dr\,d\theta &= \int_0^{2\pi}\int_0^1 \left(\frac{r^4(\cos\theta)^2}{2-r^2} + \frac{r^4(\sin\theta)^2}{2-r^2} + r^2\right)^{1/2} dr\,d\theta \\ &= \int_0^{2\pi}\int_0^1 \frac{\sqrt{2}}{\sqrt{2-r^2}}\,dr\,d\theta = 2(2-\sqrt{2}))\pi\end{aligned}$$

A third alternative is to parameterize the surface as $\vec{r}(\phi,\theta) = \sqrt{2}\sin\phi\cos\theta, \sqrt{2}\sin\phi\sin\theta, \sqrt{2}\cos\phi)$, $0 \le \phi \le \pi/4$ and $0 \le \theta \le 2\pi$. The $\vec{r}_\phi \times \vec{r}_\theta = 2\sin\phi$ and if you integrate this over the rectangle $0 \le \phi \le \pi/4$ and $0 \le \theta \le 2\pi$ one gets $2(2-\sqrt{2}))\pi$.

(16) 13. Find the flux of $F(x,y,z) = 4x\vec{i} + 4y\vec{j} + 2\vec{k}$ outward (away from the z-axis). through the surface cut from the bottom of the paraboloid $z = x^2 + y^2$ by the plane $z = 1$.

The normal to the surface is $-z_x\vec{i} - z_y\vec{j} + \vec{k}$ but this is the inward normal and so the outward normal is $\vec{N} = z_x\vec{i} + z_y\vec{j} - \vec{k}$ or $2x\vec{i} + 2y\vec{j} - \vec{k}$. The flux is therefore

$$\begin{aligned}\iint_S \vec{F}\cdot\vec{n}d\sigma &= \iint_R \vec{F}(x,y,x^2+y^2)\cdot(2x\vec{i} + 2y\vec{j} - \vec{k})\,dA \\ &= \iint_R 8x^2 + 8y^2 - 1\,dA \\ &= \int_0^{2\pi}\int_0^1 8r^2 - 1r\,dr\,d\theta = \int_0^{2\pi} 2r^4 - \frac{1}{2}r^2|_0^1\,d\theta = 3\pi\end{aligned}$$