7/12/11 Instr: Denis White Name ______________________

(9) 1. Identify and sketch the graph of the surface $4x^2 + y^2 - z^2 = 1$.

This is an elliptic hyperboloid of one sheet. The planes perpendicular to the z–axis (parallel to the xy–plane) intersect the surface in ellipses. Perpendicular to the x and y–axis teh cross sections are hyperbolas.

2. The position of a particle at time t is $\vec{r}(t) = t\ln(t+1)\vec{i} + (t+1)^{1/2}\vec{j} + (t+1)^{3/2}\vec{k}$

(12)

(a) Find the velocity and acceleration vectors.

The velocity is

$$\vec{v}(t) = \vec{r}\,'(t) = (\ln(t+1) + \frac{t}{t+1})\vec{i} + \frac{1}{2}(t+1)^{-1/2}\vec{j} + \frac{3}{2}(t+1)^{1/2}\vec{j}$$

and the acceleration is

$$\vec{a}(t) = \vec{v}\,'(t) = (\frac{1}{t+1} + \frac{(t+1)-t}{(t+1)^2})\vec{i} - \frac{1}{4}(t+1)^{-3/2}\vec{j} + \frac{3}{4}(t+1)^{-1/2}\vec{j}$$

(b) Find an equation for the tangent line to the path (in part (a)) of the particle when $t = 0$.

A tangent vector is $\vec{v}(0) = \frac{1}{2}\vec{j} + \frac{3}{2}\vec{k}$. We want the tangent line at $\vec{r}(0) = \vec{j} + \vec{k}$ and that line is The tangent line is

$$\vec{r}(t) = \vec{j} + \vec{k} + t(\frac{1}{2}\vec{j} + \frac{3}{2}\vec{k})$$

3. Solve the initial problem for $\vec{r}$ as a vector function of t if

(10)

Differential equation: $\dfrac{d\vec{r}}{dt} = te^{-t}\vec{i} + e^{-t}\vec{j} - e^{-2t}\vec{k}$

Initial Condition: $\vec{r}(0) = -\vec{i} + \vec{j} + 2\vec{k}$

We integrate:

$$\vec{r}(t) = \int te^{-t}\,dt\vec{i} + \int e^{-t}\,dt\vec{j} - \int e^{-2t}\,dt\vec{k} =$$

The first integral can be done by parts ($\int u\,dv = uv - \int v\,du$ with $u = t$, $dv = e^{-t}\,dt$)

$$\vec{r}(t) = (-te^{-t} - e^{-t})\vec{i} - e^{-t}\vec{j} + \frac{1}{2}e^{-2t}\,dt\vec{k} + \vec{C}$$

It remains to find $\vec{C}$. By the initial condition

$$-\vec{i} + \vec{j} + 2\vec{k} = \vec{r}(0) = -\vec{i} - \vec{j} + \frac{1}{2}\vec{k} + \vec{C}$$

so that $\vec{C} = 2\vec{j} + (3/2)\vec{k}$ and therefore

$$\vec{r}(t) = (-te^{-t} - e^{-t})\vec{i} + (2 - e^{-t})\vec{j} + \frac{1}{2}(e^{-2t} + 3)\,dt\vec{k}$$

(8) 4. Find the length of the curve $\vec{r} = \frac{1}{3}t^{3/2}\vec{i} + 2t\vec{j} + \frac{1}{\sqrt{3}}t^{3/2}\vec{k}$, $0 \le t \le 5$.

We need the speed. Differentiate

$$\vec{v}(t) = \vec{r}\,'(t) = \frac{1}{3}\frac{3}{2}t^{1/2}\vec{i} + 2\vec{j} + \frac{1}{\sqrt{3}}\frac{3}{2}t^{1/2}\vec{k} = \frac{1}{2}t^{1/2}\vec{i} + 2\vec{j} + \frac{\sqrt{3}}{2}t^{1/2}\vec{k}$$

so that $|\vec{r}\,'(t)| = \sqrt{t/4 + 4 + 3t/4} = (4+t)^{1/2}$. The length is therefore

$$\int_0^5 (4+t)^{1/2}\,dt = \frac{2}{3}(4+t)^{3/2}|_0^5 = \frac{2}{3}\left[(9)^{3/2} - (4)^{3/2}\right] = \frac{38}{3}$$

5. (a) Find and sketch the domain of the function $f(x,y) = \sqrt{x^2 + 4y^2 - 9}$.

(14) The domain is the set $\{(x,y) : x^2 + 4y^2 \ge 9\}$ which is the exterior of an ellipse.

(b) Find an equation for and sketch the graph of the level curve of the function $f(x,y)$ of part (a) that passes through the point (3,-1).

Since $f(3,-1)=2$ we are looking for the level curve $f(x,y)=2$ which is $x^2+4y^2=13$ which is an ellipse.

(7) 6. Find the limit. $\lim_{(x,y)\to(2,-2)} \dfrac{x+y}{x^2-y^2}$

This limit is of the form 0/0 and so we look for a cancellation in the fraction

$$\lim_{(x,y)\to(2,-2)} \frac{x+y}{x^2-y^2} = \lim_{(x,y)\to(2,-2)} \frac{x+y}{(x+y)(x-y)} = \lim_{(x,y)\to(2,-2)} \frac{1}{x-y} = \frac{1}{4}$$

(9) 7. By considering different paths of approach, show that $f(x,y)$ has no limit as $(x,y)\to(0,0)$.

$$f(x,y) = \frac{y^2+xy}{x^2+y^2}$$

Consider

$$f(r\cos\theta, r\sin\theta) = \frac{(r(\sin\theta)^2 + r\cos\theta r\sin\theta}{r^2} = (\sin\theta)^2 + \cos\theta r\sin\theta$$

and so

$$\lim_{r\to 0} f(r\cos\theta, r\sin\theta) = (\sin\theta)^2 + \cos\theta r\sin\theta$$

and this limit depends on θ which means it depends on the path of approach to (0,0) and so $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

(9) 8. Find all the second partial derivatives of $f(x,y) = \ln(x+y^2)$

There are three second distinct partial derivatives. We first need the first partials.

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{1}{x+y^2} \\
\frac{\partial f}{\partial y} &= \frac{2y}{x+y^2} \\
\frac{\partial^2 f}{\partial x^2} &= -\frac{1}{(x+y^2)^2} \\
\frac{\partial^2 f}{\partial x \partial y} &= -\frac{2y}{(x+y^2)^2} \\
\frac{\partial^2 f}{\partial y^2} &= \frac{2(x+y^2)-4y^2}{(x+y^2)^2}
\end{aligned}$$

and of course the order of differentiation for the second mixed partial derivative is inconsequential.

9. Suppose we substitute $x = u^2 - v^2$ and $y = 2uv$ in a differentiable function $w = f(x, y)$. Show that
$$\frac{1}{2}\frac{\partial w}{\partial u} = \frac{\partial f}{\partial x}u + \frac{\partial f}{\partial y}v$$
and derive a comparable expression for $\partial w/\partial v$.

(10) By the chain rule in two variables:
$$\frac{\partial w}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$$
but $\partial x/\partial u = 2u$ and $\partial y/\partial u = 2v$ so that
$$\frac{\partial w}{\partial u} = \frac{\partial f}{\partial x}2u + \frac{\partial f}{\partial y}2v$$
and dividing by 2 gives the desired expression. Similarly we have
$$\frac{\partial w}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}$$
but $\partial x/\partial v = -2v$ and $\partial y/\partial v = 2u$ so that
$$\frac{\partial w}{\partial v} = \frac{\partial f}{\partial x}(-2v) + \frac{\partial f}{\partial y}2u$$
and so we have
$$\frac{1}{2}\frac{\partial w}{\partial v} = -\frac{\partial f}{\partial x}v + \frac{\partial f}{\partial y}u$$

10. (a) Find the gradient of the function $f(x, y, z) = xy + 2yz + 3xz$.

(12) The gradient is $\nabla f = (y + 3z)\vec{i} + (x + 2z)\vec{j} + (2y + 3x)\vec{k}$

(b) Find the (directional) derivative of f at $P_0(1, 3, -1)$ in the direction of $\vec{u} = \vec{i} - \vec{j} + 2\vec{k}$.

At $P_0(1, 3, -1)$, we have $\nabla f(1, 3, -1) = -\vec{j} + 3\vec{k}$ so that the directional derivative in the direction of f in the direction $\vec{u}$ is
$$D_{\vec{u}}f(1, 3, -1) = \frac{\nabla f(1, 3, -1) \cdot \vec{u}}{|\vec{u}|} = \frac{7}{\sqrt{6}}$$