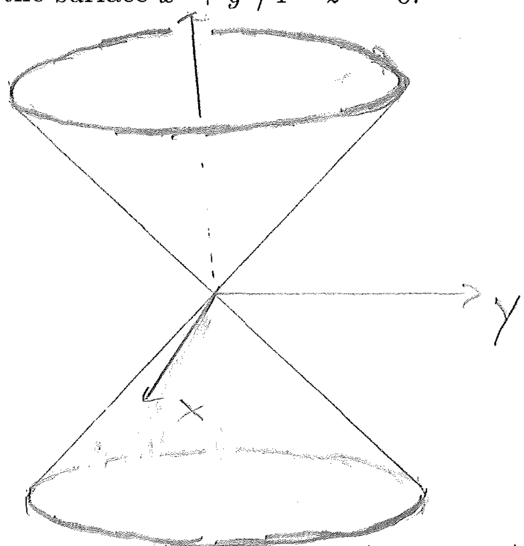
Test 1, Math 2850-005

9/29/16 Solutions Name ____________

(12) 1. Identify and sketch the graph of the surface $x^2 + y^2/4 - z^2 = 0$.

This is an elliptic cone.



(13) 2. The position of a particle at time t is $\vec{r}(t) = te^{-t}\vec{i} + 2\cos 3t\vec{j} + 2\sin 3t\vec{k}$

(a) Find the velocity.
The velocity is

$$\vec{v}(t) = \vec{r}\,'(t) = (e^{-t} - te^{-t})\vec{i} - 6\sin 3t\vec{j} + 6\cos 3t\vec{k}$$

(b) Find the unit tangent vector $\vec{T}(t)$ to the curve (in Part(a)). Then express the velocity in terms of $\vec{T}(t)$ and the speed.
We need the speed

$$|\vec{v}(t)| = \left((e^{-t} - te^{-t})^2 + (-6\sin 3t)^2 + (6\cos 3t)^2\right)^{1/2} = \left(e^{-2t}(1-t)^2 + 6\right)^{1/2}$$

so that

$$\vec{T}(t) = \frac{\vec{v}}{|\vec{v}|} = \left(e^{-2t}(1-t)^2 + 6\right)^{-1/2}\left[(e^{-t} - te^{-t})\vec{i} - 6\sin 3t\vec{j} + 6\cos 3t\vec{k}\right]$$

(c) Find parametric equations for the tangent line to the path (in part (a)) of the particle when $t = 0$.

The tangent line is in the direction of $\vec{v}(0) = \vec{i} + 6\vec{k}$ and passes through $\vec{r}(0) = 2\vec{j}$ and so has parameterization

$$\vec{R}(t) = 2\vec{j} + t(\vec{i} + 6\vec{k}) = t\vec{i} + 2\vec{j} + 6t\vec{k}$$

(9) 3. Find the length of the curve $\vec{r} = t\vec{i} + \frac{4}{3}t^{3/2}\vec{j} + t^2\vec{k}$, $0 \le t \le 5$.

We need the speed. First the velocity is $\vec{r}'(t) = \vec{i} + 2t^{1/2}\vec{j} + 2t\vec{k}$ so that the speed is $|\vec{r}'(t)| = (1+4t+4t^2)^{1/2} = ((1+2t)^2)^{1/2} = |1+2t|$ However we are interested only in $0 \le t \le 5$ and so no absolute value is needed. The length of the curve is

$$\int_0^5 |\vec{r}'(t)|\,dt = \int_0^5 1 + 2t\,dt = t + t^2|_0^5 = 30$$

(11) 4. Solve the initial problem for $\vec{r}$ as a vector function of t if

Differential equation: $\dfrac{d\vec{r}}{dt} = t\cos(t^2)\vec{i} + t\sin(t^2)\vec{j} + \dfrac{1}{t+1}\vec{k}$

Initial Condition: $\vec{r}(0) = -\frac{1}{2}\vec{i} + \vec{j} + 2\vec{k}$

We can find $\vec{r}$ by integrating except there will be a constant of integration.

$$\begin{aligned}\vec{r}(t) &= \int t\cos t^2\vec{i} + t\sin t^2\vec{j} + \frac{1}{t+1}\vec{k}\,dt \\ &= \int \frac{1}{2}\cos u\,du\vec{i} + \int \frac{1}{2}\sin u\,du\vec{k} + \ln(t+1)\vec{k} + \vec{C} \\ &= \frac{1}{2}\sin(t^2)\vec{i} - \frac{1}{2}\cos(t^2)\vec{j} + \ln(t+1)\vec{k} + \vec{C}\end{aligned}$$

where we used u–substitution with $u = t^2$ and $du = 2t\,dt$ to evaluate two of the integrals.

We can find $\vec{C}$ the constant of integration by using the equation $\vec{r}(0) = -\frac{1}{2}\vec{i} + \vec{j} + 2\vec{k}$. Substitute $t = 0$ into our expression for $\vec{r}(t)$:

$$\vec{r}(0) = -\frac{1}{2}\vec{j} + \vec{C}$$

so we must have $\vec{C} = -\frac{1}{2}\vec{i} + \frac{3}{2}\vec{j} + 2\vec{k}$ so that

$$\vec{r}(t) = \frac{1}{2}(\sin(t^2) - 1)\vec{i} - \frac{1}{2}(\cos(t^2) - 3)\vec{j} + (\ln(t+1) + 2)\vec{k}$$

(8) 5. Find the limit. $\displaystyle\lim_{(x,y)\to(1,3)} \frac{y-3x}{y^2 - 2xy - 3x^2}$

This limit is of the form 0/0 and so we look for some cancellation: Factor the bottom: $y^2 - 2xy - 3x^2 = (y-3x)(y+x)$ so that

$$\lim_{(x,y)\to(1,3)} \frac{y-3x}{y^2-2xy-3x^2} = \lim_{(x,y)\to(1,3)} \frac{y-3x}{(y-3x)(y+x)} \lim_{(x,y)\to(1,3)} \frac{1}{y+x} = \frac{1}{4}$$

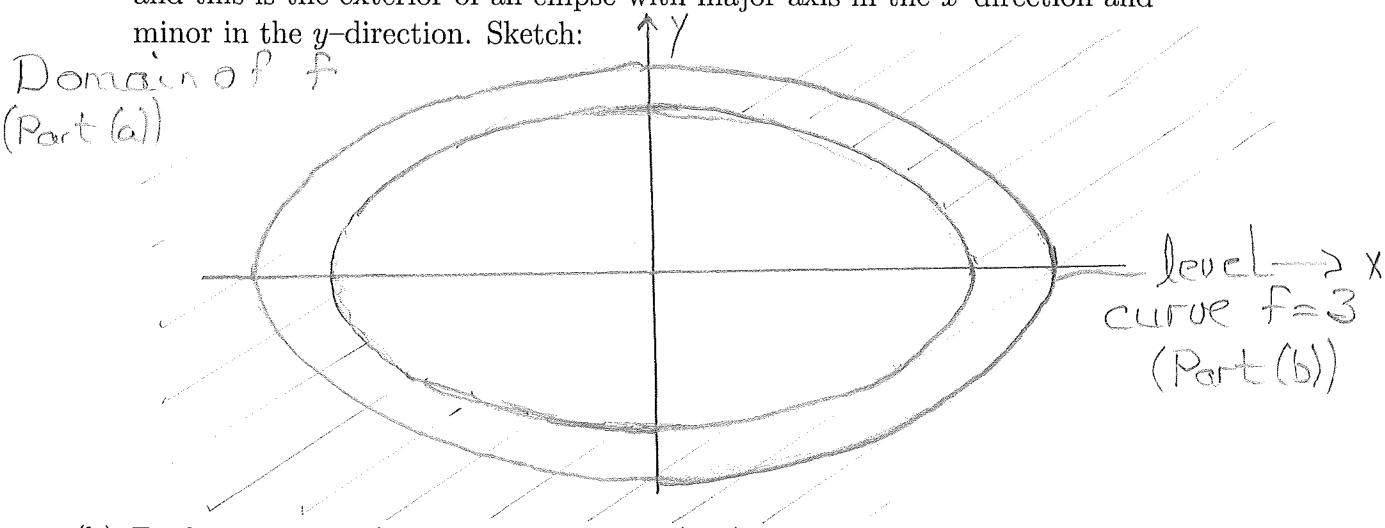
(12) 6. For the function $f(x, y) = \sqrt{x^2 + 4y^2 - 16}$.

(a) Find and sketch the domain of f.

The expression for f makes sense provided $x^2 + 4y^2 - 16 \geq 0$ so that we do not take the square root of a negative number. The domain is therefore all (x, y) so that

$$\frac{x^2}{16} + \frac{y^2}{4} \geq 1$$

and this is the exterior of an ellipse with major axis in the x–direction and minor in the y–direction. Sketch:



(b) Find an equation for the level curve of $f(x, y)$ that passes through the point (3,-2). Include a graph of that level curve in your sketch of Part (a).

Since $f(3, -2) = 3$ the level curve is $\sqrt{x^2 + 4y^2 - 16} = 3$ (that is $f = 3$) and simplifying we get

$$\frac{x^2}{25} + \frac{y^2}{25/4} = 1$$

which is an ellipse. We graph it on the graph in Part (a).

7. Suppose we substitute $x = (u+v)/\sqrt{2}$ and $y = (-u+v)/\sqrt{2}$ (which corresponds to rotating the xy-plane by 45 degrees) in a differentiable function $w = f(x, y)$.
(8) Express $\partial w/\partial u$ in terms of $\partial w/\partial x$ and $\partial w/\partial y$

By the chain rule

$$\begin{aligned}\frac{\partial w}{\partial u} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} \\ &= \frac{\partial w}{\partial x}\frac{1}{\sqrt{2}} - \frac{\partial w}{\partial y}\frac{1}{\sqrt{2}}\end{aligned}$$

(14) 8. (a) Find the gradient of the function $f(x, y) = y^2 e^{xy}$.

$$\nabla f = y^3 e^{xy}\vec{i} + (2y + xy^2)e^{xy}\vec{j}$$

(b) Find the (directional) derivative of f at $P_0(0,2)$ in the direction of $\vec{u} = \vec{\imath} - 3\vec{\jmath}$.
We need $\nabla f(0,2) = 8\vec{\imath} + 4\vec{\jmath}$ and the formula for the directional derivative gives

$$D_{\vec{u}} f(0,2) = \frac{\nabla f(0,2) \cdot \vec{u}}{|\vec{u}|} = \frac{(8\vec{\imath} + 4\vec{\jmath}) \cdot (\vec{\imath} - 3\vec{\jmath})}{\sqrt{1^2 + (-3)^2}} = \frac{-4}{\sqrt{10}}$$

(c) Find the direction $\vec{u}$ so that $D_{\vec{u}} f(0,2)$ is largest. Find the derivative in that direction.
The direction that maximizes the directional derivative is $\nabla f / |\nabla f|$ which is $(8\vec{\imath} + 4\vec{\jmath})/\sqrt{8^2 + 4^2} = (2\vec{\imath} + \vec{\jmath})/\sqrt{5}$. The derivative in that direction is just $|\nabla f| = \sqrt{80}$