Test 2, Math 2850

7-26-11 Solutions Name ____________________

(%)

1. Consider the surface $x^4 + 3xz + z^2 + \cos(\pi xy) = -2$ and the point $P_0(-1,1,2)$ on that surface. Find an equation of

(16)

(a) the tangent plane at P_0
For $F(x,y,z) = x^4 + 3xz + z^2 + \cos(\pi xy)$, we know that ∇F is perpendicular to the level surface.

$$\begin{aligned} \nabla F(x,y,z) &= (4x^3 + 3z - \sin(\pi xy)\pi y)\vec{i} - \sin(\pi xy)\pi x\vec{j} + (3x+2z)\vec{k} \\ \nabla F(-1,1,2) &= 2\vec{i} + \vec{k} \end{aligned}$$

so that the tangent plane $2(x-(-1)) + (z-2) = 0$ or $2x + z = 0$

(b) the normal line to the surface at P_0.
The normal line is $\vec{r}(t) = -\vec{i} + \vec{j} + 2\vec{k} + t(2\vec{i} + \vec{k})$

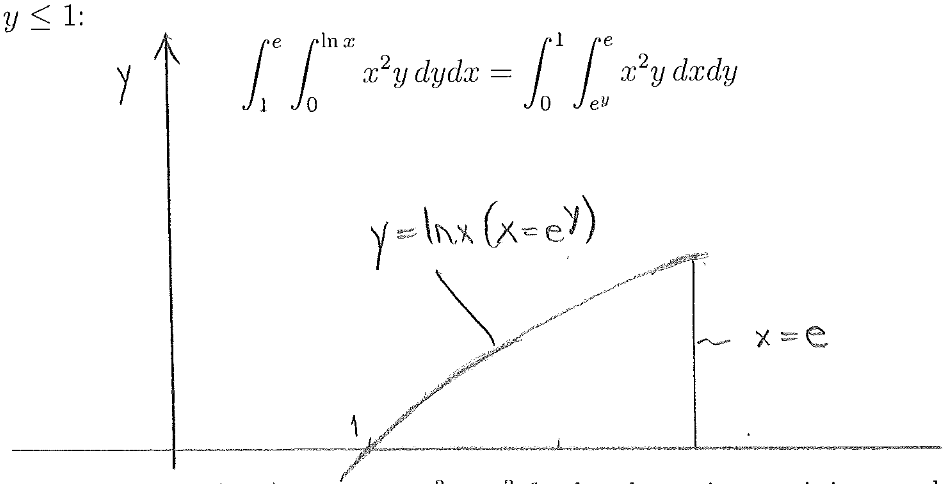
2. Sketch the region of integration and write an equivalent double integral with the the order of integration reversed. Do NOT evaluate.

(10)

$$\int_1^e \int_0^{\ln x} x^2 y \, dydx$$

The region is $0 \le y \le \ln x$, $1 \le x \le e$. Sketch. The region is also $1 \le x \le e^y$, $0 \le y \le 1$:

$$\int_1^e \int_0^{\ln x} x^2 y \, dydx = \int_0^1 \int_{e^y}^{e} x^2 y \, dxdy$$



3. Test the function $f(x,y) = 3xy - x^3 - y^3$ for local maxima, minima and saddle points.

(14)

Check for critical points. These occur when $\nabla f = \vec{0}$ or f is not differentiable but f is differentiable everywhere (because the partial derivatives exist and are continuous). Compute $\nabla f = (3y - 3x^2)\vec{i} + (3x - 3y^2)\vec{j}$. Set $\nabla f = \vec{0}$ gives $3y - 3x^2 = 0$ and $3x - 3y^2 = 0$. Therefore $y = x^2$ and $x = y^2$ so that $x = (x^2)^2 = x^4$. Therefore $x(1 - x^3) = 0$ which says $x = 0$ (in which case $y = 0$) or $x = 1$ (in which case $y = 1$). The critical points are (0,0) and (1,1).

Consider first (0,0). We have $f_{xx} = -6x$, $f_{xy} = 3$, $f_{yy} = -6y$ At (0,0), $f_{xx} = 0 = f_{yy}$ so that $f_{xx}f_{yy} - f_{xy}^2 = -9$ so that (0,0) is a saddle point by the second derivative test.

Consider next (1,1). At (1,1), $f_{xx} = -6$, $f_{yy} = -6$, so that $f_{xx}f_{yy} - f_{xy}^2 = 27$. Since $f_{xx} < 0$ this says (1,1) is a local maximum.

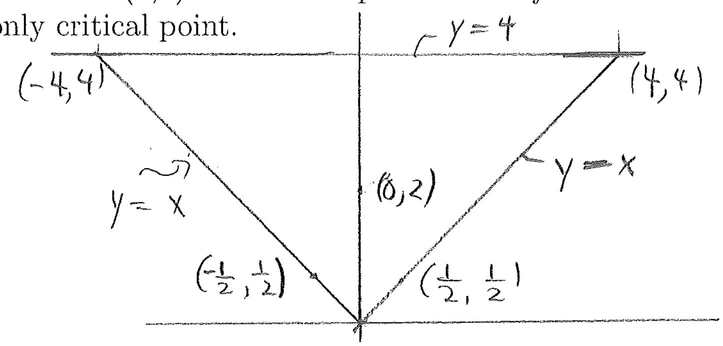
(18)

4. Find the absolute maximum and minimum values of $f(x,y) = 3x^2 + y^2 - 4y$ on the closed triangular region R in the xy–plane bounded by the lines $y = 4$ $y = x$ and $y = -x$. (Show all your work.)

Sketch the triangular region. First we check the interior for critical points. We see

$$\nabla f = 6x\vec{i} + (2y - 4)\vec{j}$$

Set $\nabla f = \vec{0}$ and we see $x = 0$ and $y = 2$ so that (0,2) is a critical point. Since f is differentiable everywhere this is the only critical point.



Next we check the edges. Consider first the line $y = 4$ $(-4 < x < 4)$ $f(x,4) = 3x^2$ and this function has a minimum at $x = 0$ (set $(d/dx)3x^2 = 0$). We get the point (0,4).

The second edge is $y = x$ $0 < x < 4$ where $f(x,x) = 3x^2 + x^2 - 4x = 4x^2 - 4x$. Check for maxima and minima. Differentiate $(d/dx)(4x^2 - 4x) = 8x - 4$. The derivative is 0 when $x = 1/2$ so that we get (1/2,1/2) is a second possible point.

The third and final edge is $y = -x$, $-4 \le x \le 0$ where $f(x,-x) = 3x^2 + x^2 + 4x$ Differentiate $(d/dx)(4x^2 + 4x) = 8x + 4$ and this is 0 when $x = -1/2$ Therefore (-1/2,1/2) is a third point.

The corners are (-4,4),(4,4) and (0,0).

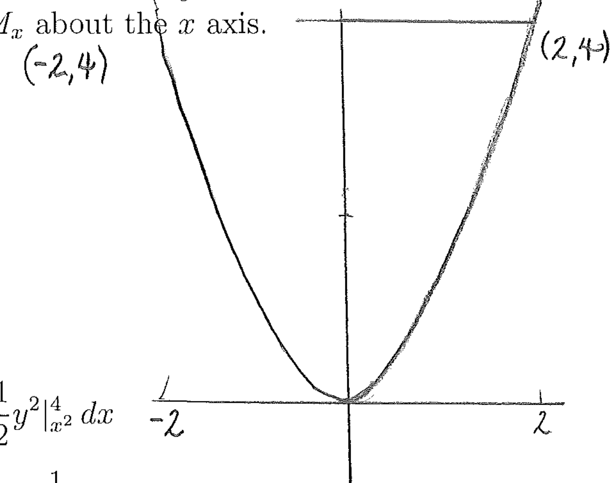
Evaluate f at all the points obtained.

Point P	f(P)
$(0,2)$	f(0,2) =-4
$(0,4)$	f(0,4)=0
$(1/2,1/2)$	f(1/2,1/2)=-1
$(-1/2,1/2)$	f(-1/2,1/2)=-1
$(-4,4)$	f(-4,4)=48
$(4,4)$	f(4,4)=48
$(0,0)$	$f(0,0) = 0$

We see that f takes its absolute maximum value of 48 at (4,4) and (-4,4) and its absolute minimum value of -4 at (0,2).

5. Find the center of mass of a thin plate of constant density δ, bounded by the parabola $y = x^2$ and the line $y = 4$. (Suggestion: it should be obvious from your picture that $\overline{x} = 0$ by symmetry. It suffices to compute $\overline{y}$. Mass is $32\delta/3$.)

(13) We sketch the region in the xy-plane. It is symmetric about the y-axis. We must compute both the mass M and the first moment M_x about the x axis.



$$M_x = \iint_R y\delta \, dA = \delta \int_{-2}^{2} \int_{x^2}^{4} y \, dy \, dx = \delta \int_{-2}^{2} \frac{1}{2} y^2 |_{x^2}^{4} \, dx$$
$$= \delta \int_{-2}^{2} 8 - \frac{1}{2} x^4 \, dx$$
$$= \delta [8x - \frac{1}{10} x^5 |_{-2}^{2}$$
$$= \delta [16 - \frac{32}{10} - (-16 + \frac{32}{10}] = \delta \frac{128}{5}$$

whereas the mass is

$$M = \iint_R \delta \, dA = \delta \int_{-2}^{2} \int_{x^2}^{4} dy \, dx = \delta \int_{-2}^{2} y |_{x^2}^{4} \, dx$$
$$= \delta \int_{-2}^{2} 4 - x^2 \, dx$$
$$= \delta [4x - \frac{1}{3} x^3 |_{-2}^{2}$$
$$= \delta [8 - \frac{8}{3} - (-8 + \frac{8}{3}] = \delta \frac{32}{3}$$

Therefore $\overline{y} = M_x / M = 12/5$. The center of mass is (0,12/5).

6. Change the Cartesian integral to an equivalent polar integral. Then evaluate the polar integral

$$\int_0^2 \int_{-\sqrt{4-x^2}}^{0} \frac{x e^{x^2+y^2}}{\sqrt{x^2+y^2}}, dy \, dx$$

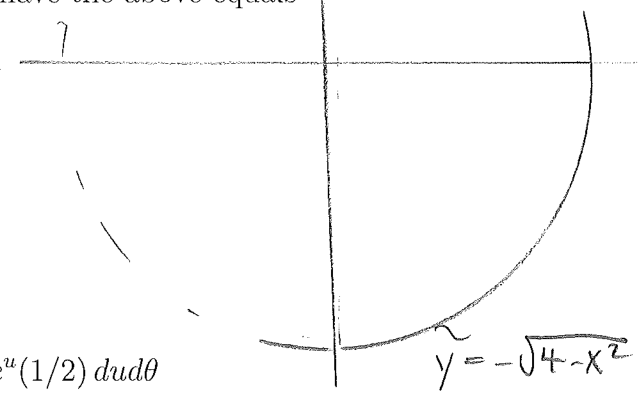
(13) The region of integration is bounded by $-\sqrt{4-x^2} \le y \le 0$ and $0 \le x \le 2$ which is a quarter of a circle ($y = -\sqrt{4-x^2}$) of radius 2 centered at (0,0) and lying

in the fourth quadrant. To convert we describe this region as $0 \le r \le 2$ and $-\pi/2 \le \theta \le 0$. We also have $x^2 + y^2 = r^2$ amd $x = r\cos\theta$ and dA is $r\,dr\,d\theta$ so that

$$\int_0^2 \int_{-\sqrt{4-x^2}}^0 \frac{xe^{x^2+y^2}}{\sqrt{x^2+y^2}}\,dy\,dx = \int_{-\pi/2}^0 \int_0^2 \frac{r\cos\theta e^{r^2}}{r} r\,dr\,d\theta$$

$$= \int_{-\pi/2}^0 \cos\theta \int_0^2 e^{r^2} r\,dr\,d\theta$$

Substitute $u = r^2$ so that $du = 2r\,dr$ and we have the above equals



$$\int_{-\pi/2}^0 \cos\theta \int_0^2 e^{r^2} r\,dr\,d\theta = \int_{-\pi/2}^0 \cos\theta \int_0^4 e^u (1/2)\,du d\theta$$

$$= \int_{-\pi/2}^0 \cos\theta (1/2) e^u|_0^4\,du d\theta$$

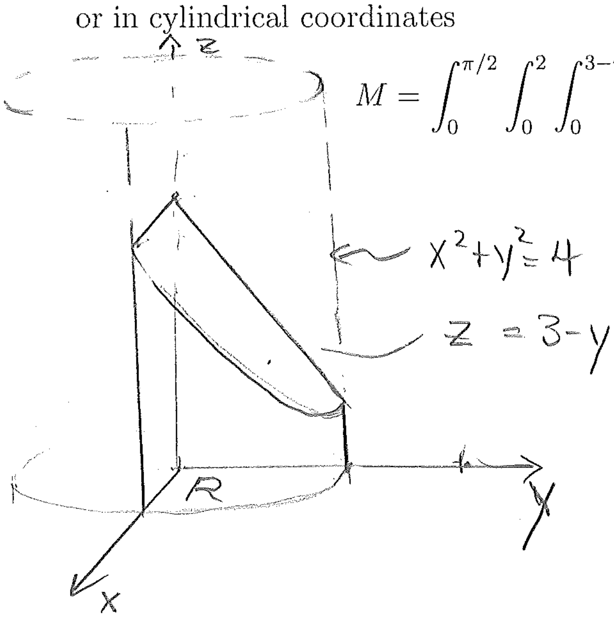
$$= \frac{e^4-1}{2} \int_{-\pi/2}^0 \cos\theta d\theta = \frac{e^4-1}{2} \sin\theta|_{-\pi/2}^0 = \frac{e^4-1}{2}$$

7. Find the mass of the solid in the first octant bounded by the coordinate planes, the plane $z + y = 3$ and the cylinder $x^2 + y^2 = 4$ if the density is $\delta(x, y, z) = x$.
(16) Sketch the solid. The mass is

$$M = \iiint_D x\,dV = \int_0^2 \int_0^{\sqrt{4-y^2}} \int_0^{3-y} x\,dz\,dx\,dy$$

or in cylindrical coordinates

$$M = \int_0^{\pi/2} \int_0^2 \int_0^{3-r\sin\theta} r\cos\theta\,dzr\,dr\,d\theta$$



R - quarter circle
of radius 2
in first quadrant.