(%) **Test 2**, Math 2850-005

11-03-16 Solutions Name ____________________

1. Consider the surface $x^4 + 2xy + y^3 + z^2 + e^{xz} = 1$ and the point $P_0(-1, 1, 0)$ on
(14) that surface. Find

   (a) the tangent plane at P_0

   The given surface is a level surface of $f(x, y, z) = x^4 + 2xy + y^3 + z^2 + e^{xz}$ We need $\nabla f = (4x^3 + 2y + ze^{xz})\vec{i} + (2x + 3y^2)\vec{j} + (2z + xe^{xz})\vec{k}$ and particularly $\nabla f(-1, 1, 0) = -2\vec{i} + \vec{j} - \vec{k}$. This vector is orthogonal to the tangent plane and so the tangent plane is

   $$-2(x + 1) + (y - 1) - z = 0 \quad \text{or} \quad -2x + y - z = 3$$

   (b) the normal line to the surface at P_0.

   The normal line has direction $-2\vec{i} + \vec{j} - \vec{k}$ and passes through P_0 and so has parameterization.

   $$\vec{r}(t) = -\vec{i} + \vec{j} + t(-2\vec{i} + \vec{j} - \vec{k})$$

2. Sketch the region of integration and write an equivalent double integral with the
(14) the order of integration reversed. Evaluate the double integral.

$$\int_0^1 \int_x^1 \frac{xe^y}{y^2}\, dy\, dx$$

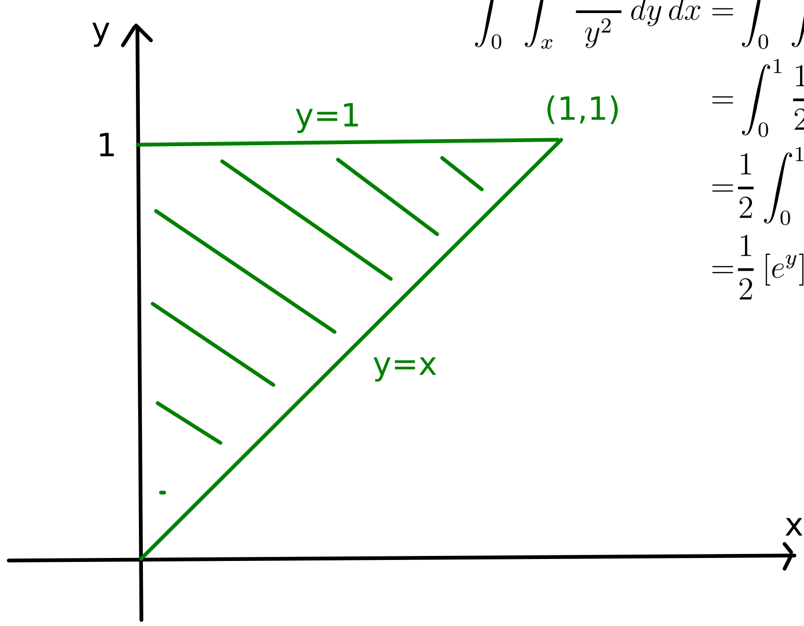
The region of integration is $\{(x, y) : x \le y \le 1, 0 \le x \le 1\}$ is a triangle (sketch!) which can be equally well be described as $\{(x, y) : 0 \le x \le y, 0 \le y \le 1\}$ and so

$$\int_0^1 \int_x^1 \frac{xe^y}{y^2}\, dy\, dx = \int_0^1 \int_0^y \frac{xe^y}{y^2}\, dx\, dy$$

$$= \int_0^1 \frac{1}{2} x^2|_0^y \frac{e^y}{y^2}\, dy$$

$$= \frac{1}{2} \int_0^1 e^y\, dy$$

$$= \frac{1}{2} \left[e^y\right]_0^1 = \frac{e - 1}{2}$$



3. Test the function $f(x, y) = 3xy - x^3 + y^3$ for local maxima, minima and saddle
(15) points.

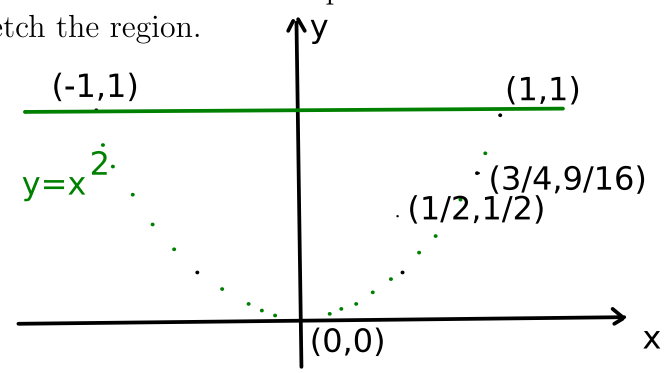
We need to find all critical points. They occur where $\nabla f = \vec{0}$ or where f is not differentiable but, as a polynomial, f is differentiable everywhere. $\nabla f = (3y - 3x^2)\vec{i} + (3x + 3y^2)\vec{j}$ and so $\nabla f = \vec{0}$ implies $3y - 3x^2 = 0$ ($y = x^2$) and $3x + 3y^2 = 0$ ($x = -y^2$) so that $y = x^2 = (-y^2)^2 = y^4$ So we have $y(1 - y^3) = 0$ which means $y = 1$ or $y = 0$. If $y = 0$ then $x = 0$ and if $y = 1$ then $x = -1$. There are two critical points (0,0) and (-1,1). Use the second derivative test to check for local max, min or saddle points. We need $f_{xx} = -6x$ $f_{xy} = 3$ and $f_{yy} = 6y$.

(0,0) At the point (0,0), $D = f_{xx}f_{yy} - (f_{xy})^2 = -9 < 0$ and so this is a saddle point.

(-1,1) At the point (-1,1), $D = (6)(6) - 3^2 = 27 > 0$ and since $f_{xx} = 6 > 0$ this is a local minimum.

4. Find the absolute maximum and minimum values of $f(x, y) = 2x^2 - 4xy + 4y^2 - 2y$ region R in the xy–plane, bounded by the line $y = 1$, the curve $y = x^2$ (Show
(16) your work.)

This is the closed bounded region method. We check the interior for critical points and the edges for local extrema and compare function values at these points plus any corners. Sketch the region.



I. Interior Find the critical points of f. We need $\nabla f = (4x - 4y)\vec{i} + (-4x + 8y - 2)\vec{j}$. Set $\nabla f = (0, 0)$ and we have $4x - 4y = 0$ and $-4x + 8y - 2 = 0$ and so $x = y$ and $4y - 2 = 0$ so that the only critical point is at $(1/2, 1/2)$.

II. Edges There are two edges.

$y = 1$ Here $f(x, 1) = 2x^2 - 4x + 4 - 2 = 2x^2 - 4x + 2$, $-1 \leq x \leq 1$. Check this function for critical points $(d/dx)f(x, 4) = 4x - 4$. The only critical point is $x = 1$: (1,1) is one more point. (It is also a corner.)

$y = x^2$ Here $f(x, x^2) = 2x^2 - 4x(x^2) + 4(x^2)^2 - 2x^2 = 4x^4 - 4x^3$,$-1 \leq x \leq 1$. Critical points for this function occur when $(d/dx)4x^4 - 4x^3 = 0$ so $16x^3 - 12x^2 = 4x^2(4x - 3) = 0$. so $x = 0$ or $x = 3/4$. We get two points (0,0) and (3/4,9/16).

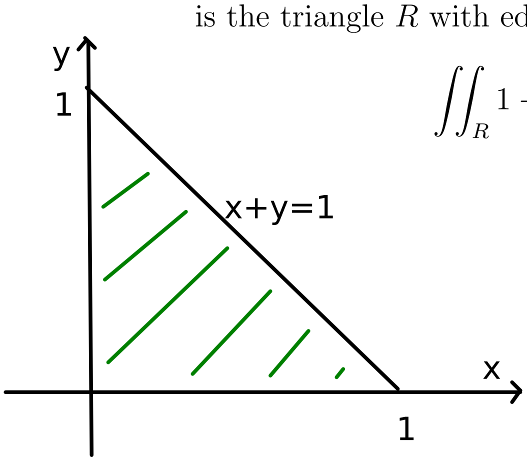
III Corners (1,1), (-1,1).

IV Evaluate $f(x,y) = 2x^2 - 4xy + 4y^2 - 2y$ at all the points found.

$$\begin{aligned}
f(1/2,1/2) =& -1/2 \quad \mathit{AbsoluteMin} \\
f(0,0) =& 0 \\
f(3/4,9/16) =& -27/64 \\
f(1,1) =& 0 \\
f(-1,1) =& 8 \quad \mathit{AbsoluteMax}
\end{aligned}$$

The absolute max value 8 occurs at (-1,1) and the absolute min value -1/2 occurs at (1/2,1/2).

5. Find the volume of the wedge cut from the first octant by the parabolic cylinder
(13) $z = 1 - x^2$ and the plane $x + y = 1$. The projection of the wedge on the xy–plane is the triangle R with edges $y = 0$, $x = 0$ and $x + y = 1$. The volume is



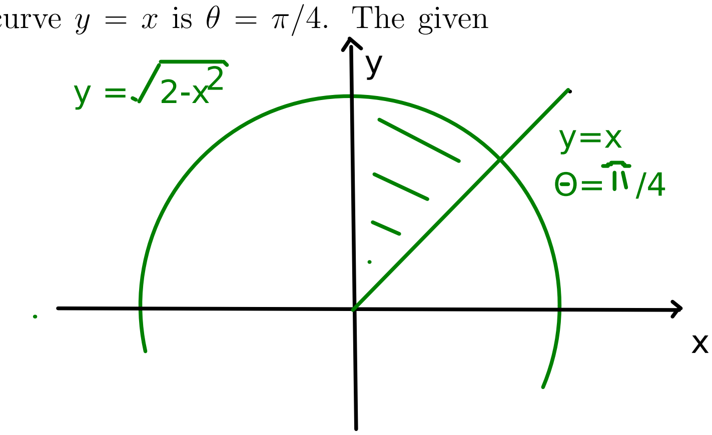
$$\begin{aligned}
\iint_R 1 - x^2\, dA =& \int_0^1 \int_0^{1-x} 1 - x^2\, dy\, dx \\
=& \int_0^1 (1 - x^2) y|_0^{1-x}\, dx \\
=& \int_0^1 1 - x - x^2 + x^3\, dx \\
=& x - \frac{1}{2}x^2 - \frac{1}{3}x^3 + \frac{1}{4}x^4|_0^1 = \frac{5}{12}
\end{aligned}$$

6. Change the Cartesian integral to an equivalent polar integral. Then evaluate the polar integral

$$\int_0^1 \int_x^{\sqrt{2-x^2}} \frac{1}{\sqrt{x^2+y^2+1}}\, dy\, dx$$

(15)

Sketch the region of integration $R = \{(x,y) : x \le y \le \sqrt{2-x^2}, 0 \le x \le 1\}$ The boundary curve $y = \sqrt{2-x^2}$ is $x^2 + y^2 = 2$, $y \ge 0$ and so it is the upper half of the circle of radius $\sqrt{2}$. The boundary curve $y = x$ is $\theta = \pi/4$. The given integral is equivalent to



$$\int_{\pi/4}^{\pi/2} \int_0^{\sqrt{2}} \frac{1}{\sqrt{r^2+1}}\, r\, dr\, d\theta$$

To evaluate this we substitute $u = r^2 + 1$ so that $du = 2r\,dr$

$$\begin{aligned}
\int_{\pi/4}^{\pi/2} \int_0^{\sqrt{2}} \frac{1}{\sqrt{r^2+1}}\, r\,dr\,d\theta &= \int_{\pi/4}^{\pi/2} \int_1^3 \frac{1}{2} u^{-1/2}\,du\,d\theta \\
&= \int_{\pi/4}^{\pi/2} u^{1/2}|_1^3\,d\theta \\
&= (3^{1/2}-1)\theta|_{\pi/4}^{\pi/2} = \frac{(3^{1/2}-1)\pi}{4}
\end{aligned}$$

7. A solid is bounded below by the plane $z = 0$, on the sides by the elliptical cylinder $x^2 + 4y^2 = 4$ and above by the plane $z = 2 - x$. Set up a (triple) iterated integral for the mass of the solid if the density is $\delta(x, y, z) = x + 3$. Do NOT evaluate the
(13) integral.

The mass is

$$\begin{aligned}
\iiint_D x + 3\,dV &= \int_{-1}^{1} \int_{-\sqrt{4-4y^2}}^{\sqrt{4-4y^2}} \int_0^{2-x} x + 3\,dz\,dx\,dy \text{ OR} \\
&= \int_{-2}^{2} \int_{-\sqrt{1-x^2/4}}^{\sqrt{1-x^2/4}} \int_0^{2-x} x + 3\,dz\,dy\,dx
\end{aligned}$$