12.6 Quadrics and Cylinder Surfaces:

Example: What is $y = x$? More correctly what is $\{(x, y, z) \in \mathbb{R}^3 : y = x\}$? It's a plane. What about $y = x^2$? Its a cylinder surface. What about $y - z = x^2$ Again a cylinder surface but, in planes parallel to the xy–plane there is a parabola that moves up the y axis as you move up (positive z direction) the parabola.

A quadric surface is a surface in $\mathbb{R}^3$ that can be described by a second order equation: $z = xy$ is an example and we see this is second order. However we shall focus on those quadrics with equation of the form

$$Ax^2 + By^2 + Cz^2 + Dz + Ey + Fx + G = 0$$

We shall work through cases beginning with the cases where A, B, C are all nonzero. In this case what is important is the relative signs of the coefficients A, B, and C.

Example: Graph and identify $4x^2 + 2y^2 + z^2 = 8$.

Solution: As an equation of three variables this will be the equation of a surface in $\mathbb{R}^3$. To discover its shape we intersect it with various planes. Lets try the plane $z = 0$ which is of course the xy-plane: the intersection of our surface withthe plane $z =$ is a curve with equation.

$$\frac{x^2}{2} + \frac{y^2}{4} = 1$$

which is an ellipse of major axis 2 and minor axis $\sqrt{2}$. Try the plane $z = 2$.

$$\frac{x^2}{1} + \frac{y^2}{2} = 1$$

and this is again an ellipse with major axis $\sqrt{2}$ and minor axis 1. This ellipse is smaller than the ellipse in the plane $z = 0$.

Next try the plane $z = -2$ and we get the same ellipse as in the case $z = 2$. But what happens if $z = 3$? $4x^2 + 2y^2 = -1$. And if $z = 2\sqrt{2}$ then $4x^2 + 2y^2 = 0$ which is a point.

We could check planes parallel to the other coordinate axes (x is constant or y is constant) but you may already see the surface

It is called and ellipsoid.

Example: Graph and identify $x^2 + y^2 - z^2 = 1$

Solution: Again we can choose planes parallel to the xy-plane: if $z = 0$ our surface intersects the xy-plane in the curve $x^2 + y^2 = 1$. More generally.

Plane	Curve of Intersection
$z = 0$	$x^2 + y^2 = 1$
$z = \pm 2$	$x^2 + y^2 = 5$
$z = \pm 5$	$x^2 + y^2 = 26$

We see therefore that the intersections with planes parallel to the xy–plane are circles and the radius increases almost proportionally to the distance from the xy–plane. To get a better grasp of the cross section of the surface we intersect it with planes parallel to the xz–plane.

Plane	Curve of Intersection
$y = 0$	$x^2 - z^2 = 1$
$y = \pm 1$	$x^2 - z^2 = 0$
$y = \pm 2$	$x^2 - z^2 = -3$
$z = \pm 5$	$x^2 - z^2 = -24$

We get hyperbolas except when $x^2 - z^2 = 0$. Graph.

This is a hyperboloid of one sheet.

Example: Graph and identify $x^2 + y^2 - z^2 = -1$

Solution: The analysis is much the same as in the previous example. One iddference is that when $z = 0$ we get $x^2 + y^2 = -1$ and that is empty. The surface does not intersect the xy–plane. Graph

This is a hyperboloid of two sheets.

Example: Graph and identify $x^2 + y^2 - z^2 = 0$

Solution: The analysis is much the same as in the previous two examples. This time when $z = 0$ we get $x^2 + y^2 = 0$. When $y = 0$ the curve of intersection (with the xz–plane) is $z = \pm x$. Graph

This is an elliptic cone.

At this stage we have exhausted all the quadric surfaces whose equation includes x^2, y^2 and z^2 terms.

Example Graph $x^2 + y^2 - z = 0$

Solution: Again we can choose planes parallel to the xy-plane: if $z = 0$ our surface intersects the xy-plane in the curve $x^2 + y^2 = 0$. More generally.

Plane	Curve of Intersection
$z = 0$	$x^2 + y^2 = 0$
$z = 4$	$x^2 + y^2 = 4$
$z = -4$	$x^2 + y^2 = -4$
$z = 25$	$x^2 + y^2 = 25$

However if we consider planes parallel to the yz–plane we get

Plane	Curve of Intersection
$x = 0$	$y^2 = z$
$x = \pm 2$	$y^2 + 4 = z$

Graph

This is an elliptic paraboloid.

Example Graph $-x^2 + y^2 - z = 0$

Solution: Again we can choose planes parallel to the xy-plane: if $z = 0$ our surface intersects the xy-plane in the curve $-x^2 + y^2 = 0$ (two straight lines). More generally.

Plane	Curve of Intersection
$z = 0$	$-x^2 + y^2 = 0$
$z = 4$	$-x^2 + y^2 = 4$
$z = 25$	$-x^2 + y^2 = 25$

However if we consider planes parallel to the yz–plane we get

Plane	Curve of Intersection
$x = 0$	$y^2 = z$
$x = \pm 2$	$y^2 + 4 = z$

Graph

This is a hyperbolic paraboloid.

Quadrics Summary

Identify quadric surfaces of the form: $Ax^2 + By^2 + Cz^2 + Dx + Ey + Fz = G$

1. **All of A, B and C are Nonzero:**

   Here D, E and F correspond to translations (for complete the square) and do not change the shape or orientation of the surface and so let's take them all to be zero. Therefore we can write if $G \neq 0$

   $$\pm \frac{x^2}{a^2} \pm \frac{y^2}{b^2} \pm \frac{z^2}{c^2} = 1$$

   (a) **Three +'s:** Ellipsoid.

   (b) **Two +'s:** Elliptic Hyperboloid of 1 Sheet.

   (c) **One +:** Elliptic Hyperboloid of 2 Sheets.

2. **If** $G = 0$

   $$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$$

   which is an Elliptic Cone. (The right side is 0 and there are 2 +'s and one − or 2 −'s and one +.

3. **One of x^2, y^2 or z^2 is Missing:**

   Again if there is a quadratic term then we can ignore the corresponding linear term: for example of $A \neq 0$ then D just corresponds to a translation (for complete the square) and does not change the shape so assume $D = 0$. Similarly the constant G corresponds to a translation in the variable not squared. If z^2 is missing then the equation can be chosen to be of the form.

   $$\pm \frac{x^2}{a^2} \pm \frac{y^2}{b^2} - z = 0$$

   (a) **Two +'s or Two −'s:** Elliptic Paraboloid.

   (b) **One + and One −:** Hyperbolic Paraboloid (or saddle surface).

4. The other surfaces are cylinder surfaces or degenerate surfaces (like planes). For example $z = y^2$ is a huge trough with parabolic cross section sitting on the x-axis at

Examples: (Answers below.)

1. $x^2 + 2y^2 + 3z^2 = 36$
2. $x^2 + 2y^2 + 3z^2 - 12y = 18$
3. $8x - y^2 + 4z^2 = 0$
4. $x^2 - y^2 + 9z^2 = 0$

Answers:

1. Ellipsoid
2. Ellipsoid (same as above but translated 3 units in the positive y direction.
3. Hyperbolic Paraboloid
4. Elliptic Cone ("centered" on the y axis.)