13.1 Curves in Space:

We are interested in subsets of $\mathbb{R}^3$ (such as quadrics studied in the previous section) and also functions defined on or to $\mathbb{R}^3$. We begin with mappings $\vec{r}(t)$ which describe position at time t of a particle, rocket or car. $\vec{r}$ is a function defined on a real interval and taking values in $\mathbb{R}^3$.

Example: What is

$$\vec{r}(t) = (t+1)\vec{i} + (2t-1)\vec{j} - 3t\vec{k} = (\vec{i}-\vec{j}) + t(\vec{i}+2\vec{j}-3\vec{k})?$$

The path passes through $\vec{i}-\vec{j}$ (which is the point (1,-1,0)) at time $t=0$ and proceeds in the direction $\vec{i}+2\vec{j}-3\vec{k}$ and reaches (2,1,-3) by time $t=1$. As time varies this curve traces out a straight line through (1,-1,0) with direction vector $\vec{i}+2\vec{j}-3\vec{k}$.

Example: Find an equation for the straight line through (1,-3,2) and (4,1,1).

Solution: There are many many solutions but

$$\vec{r}(t) = \vec{i} - 3\vec{j} + 2\vec{k} + t(3\vec{i}+4\vec{j}-\vec{k})$$

works because $\vec{r}(0)$ is (1,-3,2) and $\vec{r}(1)$ is (4,1,1).

Example: Graph the vector function $\vec{r}(t) = 2\cos t\vec{i} + 3\sin t\vec{j} + t\vec{k}$.

Solution: Recall from Section 11.1 (Parameterizatin of Planar curves) what $x(t) = 2\cos t$ and $y = 3\sin t$ is. (Note if we project $\vec{r}(t) = 2\cos t\vec{i} + 3\sin t\vec{j} + t\vec{k}$ onto the xy–plane we get $(x(t), y(t)) = (2\cos t, 3\sin t)$.) This we found was an ellipse

$$\left(\frac{x(t)}{2}\right)^2 + \left(\frac{y(t)}{3}\right)^2 = (\cos t)^2 + (\sin t)^2 = 1$$

(The major axis is 3 and the minor is 2.) Moreover the ellipse is traces out counterclockwise. It follows that our curve $\vec{r}(t)$ lies above that ellipse. One can see that as time passes the curve proceeds to rise one unit distance per unit time. The curve is a spiral above the ellipse; alternatively it is a spiral on the elliptic cylinder $x^2/4 + y^2/9 = 1$.

Definition: We say that the limit of $\vec{r}(t)$ as t approaches t_0 exists if there exists $\vec{L}$ so that $|\vec{r}(t) - \vec{L}|$ can be made arbitrarily small by choosing $|t-t_0|$ sufficiently small. We write

$$\lim_{t\to t_0} \vec{r}(t) = \vec{L}$$

Unfold the notation: $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$. ($\vec{r}(t)$ has components three real valued functions.) Also $\vec{L} = L_1\vec{i} + L_2\vec{j} + L_3\vec{k}$ so that the definition says that $\lim_{t\to t_0}\vec{r}(t) = \vec{L}$ if and only if $((f(t)-L_1)^2 + (g(t)-L_2)^2 + (h(t)-L_3)^2)^{1/2}$ can be made small by choosing $|t-t_0|$ sufficiently small. It is not difficult to see that the only way to make $((f(t)-L_1)^2 + (g(t)-L_2)^2 + (h(t)-L_3)^2)^{1/2}$ small is to make $|f(t)-L_1|$, $|g(t)-L_2|$ and $|z(t)-L_3|$ all small. Therefore the limit exists if and only if all the components limits exist: $\lim_{t\to t_0}\vec{r}(t) = \vec{L}$ if and only if $\lim_{t\to t_0} x(t) = L_1$, $\lim_{t\to t_0} y(t) = L_2$ and $\lim_{t\to t_0} z(t) = L_3$. In brief we can work component by component.

Example:

$$\lim_{t\to 0}(t^2+3)\vec{i} + \frac{\sin t}{t}\vec{j} + \frac{t^2-1}{t-1}\vec{k} = 3\vec{i} + \vec{j} + \vec{k}$$

Definition A vector function $\vec{r}(t)$ is continuous at t_0 if

1. t_0 is in the domain of $\vec{r}$.
2. $\lim_{t\to t_0} \vec{r}(t) = \vec{L}$ exists.
3. $\vec{r}(t_0) = \vec{L}$.

We say $\vec{r}$ is continuous if it is continuous at every point in its domain.

It follows that $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$ is continuous if and only if x, y and z are.

Definition A vector function $\vec{r}(t)$ is differentiable at t_0 if

$$\lim_{h\to 0} \frac{\vec{r}(t_0+h) - \vec{r}(t_0)}{h} = \vec{r}'(t_0)$$

exists. The limit $\vec{r}'(t_0)$ (if it exists) is the derivative of $\vec{r}$ at t_0. We say $\vec{r}$ is differentiable if it is differentiable at every point $t - 0$ of its domain.

It follows that $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$ is differentiable if and only if x, y and z are and

$$\vec{r}'(t) = x'(t)\vec{i}'_y(t)\vec{j} + z'(t)\vec{k}$$

Example: If $\vec{r}(t) = 2t\cos t\vec{i} + 3t\sin t\vec{j} + t^2\vec{k}$ then $\vec{r}'(t) = (2\cos t - 2t\sin t)\vec{i} + (3\sin t + 3t\cos t)\vec{j} + 2t\vec{k}$

Physical Intepretation: What is

$$\frac{\vec{r}(t_0+h) - \vec{r}(t_0)}{h}?$$

It is the displacement $\vec{r}(t_0+h) - \vec{r}(t_0)$ over the time interval t_0 to t_0+h divided by the time elapsed h and that is average velocity (a vector!) Thus $\vec{r}'(t_0)$ is the instantaneous velocity and it is a vector tangent to the curve.

Picture

We therefore define the velocity of a path $\vec{r}(t)$ is $\vec{v}(t) = \vec{r}'(t)$ and the acceleration is $\vec{a}(t) = \vec{v}'(t) = \vec{r}''(t)$. Newton's Second Law is that force is $\vec{F} = m\vec{a}$ where m denotes the mass of the object being acted on.

Example: If $\vec{r}(t) = 2\cos t\vec{i} + 3\sin t\vec{j} + t\vec{k}$ (which is a spiral on an elliptic cylinder) then $\vec{v}(t) = -2\sin t\vec{i} + 3\cos t\vec{j} + \vec{k}$ and $\vec{a}(t) = -2\cos t\vec{i} - 3\sin t\vec{j}$. Notice the acceleration is toward the center of the cylinder.

Differentiation Rules: The rules for differentiation follwo from the rules for differentiating the real valued components but we mention

Rule 5: $\frac{d}{dt}(\vec{u}\cdot\vec{v})(t) = \vec{u}'(t)\cdot\vec{v}(t) + vecu(t)\cdot\vec{v}'(t)$.

Rule 6: $\frac{d}{dt}(\vec{u}\times\vec{v})(t) = \vec{u}'(t)\times\vec{v}(t) + vecu(t)\times\vec{v}'(t)$.

Example: If the speed ($|\vec{v}(t)|$) is constant then the acceleration is perpendicular to the velocity.

We know that $c^2 = |\vec{v}(t)|^2 = \vec{v}(t) \cdot \vec{v}(t)$ where c is the constant speed. Differentiate

$$0 = \vec{v}'(t) \cdot \vec{v}(t) + \vec{v}(t) \cdot \vec{v}'(t) = 2\vec{a}(t) \cdot \vec{v}(t)$$

and this just says the acceleration is perpendicular to the velocity.