13.2 Curves in Space: Integration

We define the integral of $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$.

$$\begin{aligned}\int \vec{r}(t)\,dt &= \int x(t)\,dt\vec{i} + \int y(t)\,dt\vec{j} + \int z(t)\,dt\vec{k} \\ \int_a^b \vec{r}(t)\,dt &= \int_a^b x(t)\,dt\vec{i} + \int_a^b y(t)\,dt\vec{j} + \int_a^b z(t)\,dt\vec{k}\end{aligned}$$

Example: Suppose that a particle's velocity is $\vec{v}(t) = e^t\vec{i} - te^t\vec{j} + \sqrt{t}\vec{k}$. What is its position?

We are given $\vec{r}' = \vec{v}$ and we want $\vec{r}$: the position is

$$\vec{r}(t) = \int \vec{v}(t)\,dt = e^t\vec{i} + (-te^t + e^t)\vec{j} + \frac{2}{3}t^{3/2}\vec{k} + \vec{C}$$

Notice that the constant of integration is a vector (there are 3 constants of integration).

Recall integration by parts: $u = t$, $dv = e^t\,dt$ so that $du = dt$ and $v = e^t$ and $\int u\,dv = uv - \int v\,du = te^t - e^t + C_2$.

Example: (continued) Suppose that a particle's velocity is $\vec{v}(t) = e^t\vec{i} - te^t\vec{j} + \sqrt{t}\vec{k}$. What is its position 2 seconds later if the particle starts at (3,1,2)?

Solution: We want

$$\begin{aligned}\vec{r}(2) - \vec{r}(0) &= \int_0^2 \vec{v}(t)\,dt = e^t\vec{i} + (-te^t + e^t)\vec{j} + \frac{2}{3}t^{3/2}\vec{k}|_0^2 \\ \vec{r}(2) - (3\vec{i} + \vec{j} + 2\vec{k}) &= e^2\vec{i} + (-2e^2 + e^2)\vec{j} + \frac{2}{3}2^{3/2} - (\vec{i} + \vec{j})\end{aligned}$$

and we can solve for $\vec{r}(2) = (e^2 + 2)\vec{i} - e^2\vec{j} + (\frac{2}{3}2^{3/2} + 2)\vec{k}$.

Example: A projectile has initial position $\vec{r}_0$ and initial position $\vec{v}_0$. Find the path.

Solution: We know $\vec{r}''(t) = -g\vec{k}$ where g is the gravitational constant. Integrating we find

$$\vec{r}'(t) = -gt\vec{k} + \vec{C}$$

But $\vec{r}'(0) = \vec{v}_0$ and so $\vec{v}_0 = -g(0)\vec{k} + \vec{C}$ and so

$$\vec{r}'(t) = -gt\vec{k} + \vec{v}_0$$

Integrating a second time we have

$$\vec{r}(t) = -\frac{g}{2}t^2\vec{k} + \vec{v}_0 t + \vec{C}$$

Setting $t = 0$, we have $\vec{r}_0 = \vec{r}(0) = \vec{0} + \vec{C}$ and so

$$\vec{r}(t) = -\frac{g}{2}t^2\vec{k} + \vec{v}_0 t + \vec{r}_0$$

The textbook considers the case that $\vec{v}_0 = v_0(\cos\alpha\vec{i} + \sin\alpha\vec{j})$ (and there is no $\vec{k}$.)