13.3 Arc Length

The distance traveled by a particle $\vec{r}(t) = x(t)\vec{i}+y(t)\vec{j}+z(t)\vec{k}$ in a time interval $t_1 \leq t < t_2$ is approximately $|\vec{v}(t_1)|(t_2 - t_1)$ (speed times time elapsed.) To get the distance traveled over a long time interval $a \leq t \leq b$ one adds up the approximate distances

$$|\vec{v}(t_1)|(t_2 - t_1) + |\vec{v}(t_2)|(t_3 - t_2) + |\vec{v}(t_3)|(t_4 - t_3) + \cdots + |\vec{v}(t_{n-1})|(t_n - t_{n-1})$$

where $a = t_1 < t_2 < \ldots < t_n$. In the limit as n goes to infinity and the intervals get shorter we get the distance traveled is

$$\int_a^b |\vec{v}(t)|dt = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\, dt$$

Example Find the distance traveled by the particle on the path

$$\vec{r}(t) = e^t \cos t\vec{i} + e^t \sin t\vec{j} + e^t\vec{k}$$

$0 \leq t \leq 4\pi$

Solution: Compute the speed. First we need the velocity.

$$\vec{r}\,'(t) = (e^t \cos t - e^t \sin t)\vec{i} + (e^t \sin t + e^t \cos t)\vec{j} + e^t\vec{k} = e^t[(\cos t - \sin t)\vec{i} + (\sin t + \cos t)\vec{j} + \vec{k}]$$

Therefore the speed is

$$|\vec{r}\,'(t)| = e^t[(\cos t - \sin t)^2 + (\sin t + \cos t)^2 + 1]^{1/2} = e^t[(\cos t)^2 + (\sin t)^2 - 2\cos t \sin t + (\sin t)^2 + (\cos t)^2 + 2\cos t$$

The length of the curve is

$$\int_0^{4\pi} |\vec{v}(t)|dt = \int_0^{4\pi} \sqrt{3}e^t\, dt = \sqrt{3}e^t|_0^{4\pi} = \sqrt{3}(e^{4\pi} - 1)$$

Arc Length Parameterization Given a curve $\vec{r}(t)$, $a \leq t \leq b$ and t_0 where $a \leq t_0 \leq b$ The arclength parameter with base point $\vec{r}(t_0)$ is

$$s(t) = \int_{t_0}^t |\vec{v}(\tau)|\, d\tau = \int_{t_0}^t \sqrt{(x'(\tau))^2 + (y'(\tau))^2 + (z'(\tau))^2}\, d\tau$$

which is the distance traveled along the curve in the time interval $t_0 < \tau \leq t$ Given t, s is determined but assuming $|\vec{v(\tau)}| \neq 0$ for any τ then it is also true that s determines t. (The odometer reading tells us what time it is.)

Example In the previous example $s = \int_{t_0}^t |\vec{v}(\tau)\, d\tau = \sqrt{3}\int_{t_0}^t e^\tau\, d\tau = \sqrt{3}[e^t - e^{t_0}]$. Therefore

$$t(s) = \ln[\frac{s}{\sqrt{3}} + e^{t_0}]$$

This tells us the time when we have traveled s units along the curve. Note when $s = 0$ $t(0) = t_0$. We can use this computation to "parameterize $\vec{r}$ according to arclength"

$$\begin{aligned}\vec{r}_1(s) &= e^{t(s)}\cos t(s)\vec{i} + e^{t(s)}t(s)\sin t(s)\vec{j} + e^{t(s)}\vec{k} \\ &= \left[\frac{s}{\sqrt{3}} + e^{t_0}\right]\left[\cos(\ln(\frac{s}{\sqrt{3}} + e^{t_0}))\vec{i} + \sin(\ln(\frac{s}{\sqrt{3}} + e^{t_0}))\vec{j} + \vec{k}\right]\end{aligned}$$

This is the same path but it is parameterized to that the speed as on traverses the path is one

$$\begin{aligned}\vec{r}_1'(s) &= \left[\frac{1}{\sqrt{3}}\cos(\ln(\frac{s}{\sqrt{3}}+e^{t_0}))-\sin(\ln(\frac{s}{\sqrt{3}}+e^{t_0}))\frac{1}{\sqrt{3}}\right]\vec{i}\\ &+\left[\frac{1}{\sqrt{3}}\sin(\ln(\frac{s}{\sqrt{3}}+e^{t_0}))+\cos(\ln(\frac{s}{\sqrt{3}}+e^{t_0}))\frac{1}{\sqrt{3}}\right]\vec{j}+\frac{1}{\sqrt{3}}\vec{k}\end{aligned}$$

and one can check that $|\vec{r}_1{}'(s)| = 1$.