14.1 Functions of several variables.

Definition: Suppose that $\vec{x}_0$ is a point in $\mathbb{R}^3$ (or $\mathbb{R}^n$) and $r > 0$. The open ball of radius r and center $\vec{x}$ is $B_{\vec{x}_0}(r) = \{\vec{x} : |\vec{x} - \vec{x}_0| < r\}$. $B_{\vec{x}_0}(r)$ is a disk in $\mathbb{R}^2$ and a ball in $\mathbb{R}^3$.

Suppose now that R is some set and $\vec{x}_0$ is in R. Then $\vec{x}_0$ is an *interior point* of R if there exists $r > 0$ so that $B_{\vec{x}_0}(r)$ is entirely contained in R. The set of all interior points of R is called the *interior* of R.

Now suppose that $\vec{x}_0$ is any point and not necessarily in R. Then $\vec{x}_0$ is a boundary point of R if, for every $r > 0$, $B_{\vec{x}_0}(r)$ contains points both in R and not in R. The set of all boundary points of r is the *boundary* of R

A set R that consists entirely of interior points is said to be *open*. A set R that contains its boundary is said to be *closed*.

No point can be an interior point of R and a boundary point of R. However every point of R is either an interior point or a boundary point. If the set R is an interval (in $\mathbb{R}$) then the boundary is the endpoints and the interior is all the other points in R. Also $B_{\vec{x}_0}(r) = \{\vec{x} : |\vec{x} - \vec{x}_0| < r\}$ is open but $\{\vec{x} : |\vec{x} - \vec{x}_0| \leq r\}$is closed. When introducing differentiability of a function of several variables at a point $\vec{x}_0$ we will want of our functions to be defined in an open set containing $\vec{x}_0$.

Definition: A real valued function f defined on a *domain* $D \subseteq \mathbb{R}^3$ is rule that assigns to every $\vec{x} \in D$ one and only one real number denoted $f(\vec{x})$. (Note $\vec{x} = (x_1, x_2, x_3)$.) The *range* of f is defined to be $\{w \in \mathbb{R} : w = f(\vec{x}),\ \text{for some } \vec{x} \in D\}$

Example: $f(x, y, z) = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}$ is the distance of a point $(x, y, z) \in D = \mathbb{R}^3$ to a fixed point (a, b, c).

Example: $f(x, y, z) = \ln(z - x^2 + y^2)$ has domain $D = \{(x, y, z) : z > x^2 + y^2)$ which is the region above the circular paraboloid. This is the maximal domain on which f makes sense.

Recall that the graph of a function $f(x)$ of one variable is $\{(x, y) \in \mathbb{R}^2 : y = f(x)\}$. In other words, it requires the xy plane to graph a function of one variable. For functions of two variables $f(x, y)$ the graph $\{(x, y, z) \in \mathbb{R}^3 : z = f(x, y)\}$ is in $\mathbb{R}^3$. For functions of $f(x, y, z)$ of three variables four dimensions is needed for the graph and so the graph can't serve as a visual aid. Level curves and surfaces can serve as visual aids.

Level curves: If $f(x, y)$ is a function and c is any constant then the equation $\{(x, y) : c = f(x, y)\}$ is a level curve of the f.

Example: Consider $f(x, y) = 4x^2 + y^2$. Sketch several representative level curves and then graph the function.

Solution: Consider the level curve corresponding to various values of f (and so various horizontal planes intersection with the graph of f. For example when $c = f = 1$ we have $1 = 4x^2 + y^2$ which is an ellipse with major axis 1 (on the y–axis) and minor 1/2. When $c = f = 4$ then the level curve is $4 = 4x^2 + y^2$ and the major axis is 2 and the minor is 1 and so the level curve in this case is twice as big (but the f values is 4 times as big). When $c = f = 16$, the level curve is $16 = 4x^2 + y^2$ and so the major axis is 4 and minor is 2. If we choose $c = f = 0$ then the level curve degenerates to a point (0,0) and if $c < 0$ the level "curve" is the empty set (there are no points satisfying the equation and that corresponds to the fact that f never takes negative values. In the sketch below several of the level curves are sketched in the xy–plane and labeled with the corresponding values of c.

Of course the graph of this function is the surface $z = 4x^2 + y^2$ which is an elliptic parabola with vertex the origin and opening up along the positive z–axis. We will learn later that f increases and decreases the fastest in the directions perpendicular to the level curves.

Example: If $f(x, y, z) = 9x^2 + y^2 + 4z^2$ then we can not expect to graph f because the graph ($w = f(x, y, z)$) is four dimensional. Now we are reliant on the level "surfaces" to visualize the function. If we choose $c = f = 9$ for example we see we get the level surface is an ellipsoid centered at the origin (0,0,0) and touching the coordinate axes at $(\pm 1, 0, 0)$, $(0, \pm 3, 0)$ and $(0, 0, \pm 3/2)$. If we choose $c = f = 36$ we get an ellipsoid exactly twice the size in the three directions. If we choose $c = 0$ then the level "surface" degenerates to a point (0,0,0)and the surfaces corresponding to $c < 0$ are empty (contain no points). The level surfaces can be graphed in 3-space ($\mathbb{R}^3$).

Example: If $f(x, y, z) = x^2 + y^2 - z^2$. The level surfaces in this case are $c = f = 1$ is an elliptic (circular) hyperboloid of one sheet and if $c = f = 0$ is a circular cone ("inside" the previous hyperboloid) and if $c = f = -1$ is a circular hyperboloid of two sheets ("inside" the cone) The level surfaces are sketched below. Pick a point (x_0, y_0, z_0) and use the level surfaces to see in what direction f increases (or decreases) the most rapidly. It is in the direction perpendicular to the level surfaces.