14.2 Limits and Continuity:

We now extend the concepts of limits and continuity to the case that f is a real valued function defined on a set D in $\mathbb{R}^2$ (or $\mathbb{R}^3$).

Definition: We say that the *limit* of $f(x,y)$ as (x,y) approaches (x_0,y_0) is L if $|f(x,y)-L|$ can be made arbitrarily small by choosing $0<|(x,y)-(x_0,y_0)|$ small (but not 0). We write

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y)=L.$$

The subtlety here, that did not arise in functions of a single variable, is that $f(x,y)$ must get close to L no matter how (x,y) goes to (x_0,y_0). For example it is quite possible that $\lim_{x\to x_0} f(x,y_0)$ exists but $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ does not.

Examples:

$$\lim_{(x,y)\to(1,3)} \sin x+\frac{y}{x-2}=\sin 1-3$$

$$\lim_{(x,y)\to(2,2)} \frac{x^2-y^2}{y-x}=\lim_{(x,y)\to(2,2)} -(x+y)=-4$$

$$\lim_{(x,y)\to(2,0)} \frac{x\sin y}{y}$$

Counterexample:

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} \text{ does not exist}$$

The reason is that if we approach the origin along the straight line $y=mx$ (where m is the slope) then $xy/(x^2+y^2)=mx^2/(x^2+m^2x^2)=m/(1+m^2)$ and so the limiting value depends on th epath of approach and that means there is no limit as $(x,y)\to(0,0)$.

Curvature: The curvature is a measure of how fast a curve turns. It is the reciprocal of the radius of the "osculating" circle. The curvature of the curve $\vec{r}$ at $\vec{r}(t)$ is

$$\kappa(t)=\frac{|\vec{v}(t)\times\vec{a}(t)|}{|\vec{v}|^3}$$

Example: Consider $\vec{r} = \cos 2\pi t\vec{i} + \sin 2\pi t\vec{j} + t\vec{k}$. Then

$$
\begin{aligned}
\vec{v}(t) &= -2\pi \sin 2\pi t\vec{i} + 2\pi \cos 2\pi t\vec{j} + \vec{j} \\
\vec{T}(t) &= \frac{\vec{v}(t)}{\sqrt{4\pi^2+1}} \\
\vec{a}(t) &= -4\pi^2 \cos 2\pi t\vec{i} - 4\pi^2 \sin 2\pi t \\
\vec{N}(t) &= -\cos 2\pi t\vec{i} - \sin 2\pi t \\
\vec{B}(t) &= \frac{1}{\sqrt{4\pi^2+1}} \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2\pi \sin 2\pi t & 2\pi \cos 2\pi t & 1 \\ -\cos 2\pi t & -\sin 2\pi t & 0 \end{bmatrix} \\
&= \frac{1}{\sqrt{4\pi^2+1}} \left[\sin 2\pi t\vec{i} - \cos 2\pi t\vec{j} + 2\pi\vec{k} \right] \\
\kappa &= \frac{|\vec{v} \times \vec{a}|}{|\vec{v}(t)|^3} \\
&= \frac{4\pi^2\sqrt{4\pi^2+1}}{(\sqrt{4\pi^2+1})^3} = \frac{4\pi^2}{4\pi^2+1}
\end{aligned}
$$