**14.3 Partial Derivatives**:

The partial derivative of $f(x,y)$ with respect to $x$ is the derivative of the function $x \mapsto f(x,y)$ where $y$ is regarded as a constant. It is denoted

$$\frac{\partial f}{\partial x}(x,y)$$

**Example** Find the partial derivatives of $f(x,y)$ with respect to $x$ and $y$ if

$$f(x,y) = 3x + 5y + x^2 + y^4 + 7x^2y$$

**Solution**

$$\frac{\partial f}{\partial x}(x,y) = 3 + 2x + 14xy \qquad \frac{\partial f}{\partial y}(x,y) = 5 + 4y^3 + 7x^2$$

**Example** Find the partial derivatives of $f(x,y) = y^3\cos(xy)$ with respect to $x$ and $y$ if

**Solution**

$$\frac{\partial f}{\partial x}(x,y) = -y^3\sin(xy)y = -y^4\sin(xy) \qquad \frac{\partial f}{\partial y}(x,y) = 3y^2\cos(xy) - y^3\sin(xy)x = 3y^2\cos(xy) - xy^3\sin(xy)$$

**Physical interpretation**: Consider the graph $z = f(x,y)$. If the domain of $f$ is $D \subseteq \mathbb{R}^2$ (in the $xy$-plane) then the graph is a surface in $\mathbb{R}^3$ above $D$. Then

$$\frac{\partial f}{\partial x}(x_0,y_0)$$

is the slope of the curve formed by the intersection of the surface $z = f(x,y)$ with the plane $y = y_0$, that is $z(x) = f(x,y_0)$ at the point $x = x_0$. See the picture:

and a similar interpretation applies to $\frac{\partial f}{\partial y}(x_0,y_0)$.

**Higher Order Partials** For $f(x,y,z) = e^{2x-y} + x^3y^2 + \ln xyz$ find all the second partial derivatives.

**Solution** First we need the first partials. We should use the identity $\ln xyz = \ln x +$

$\ln y + \ln z$. We introduce an alternate notation .

$$\frac{\partial f}{\partial x} = f_x = 2e^{2x-y} + 3x^2y^2 + \frac{1}{x}$$
$$\frac{\partial f}{\partial y} = f_y = -e^{2x-y} + 2x^3y + \frac{1}{y}$$
$$\frac{\partial f}{\partial z} = f_z = \frac{1}{z}$$

Now the second partials.

$$\frac{\partial^2 f}{\partial x^2} = f_{xx} = 4e^{2x-y} + 6xy^2 - \frac{1}{x^2}$$
$$\frac{\partial^2 f}{\partial y\partial x} = f_{xy} = -2e^{2x-y} + 6x^2y$$
$$\frac{\partial^2 f}{\partial z\partial x} = f_{xz} = 0$$
$$\frac{\partial^2 f}{\partial y^2} = f_{yy} = e^{2x-y} + 2x^3 - \frac{1}{y^2}$$
$$\frac{\partial^2 f}{\partial x\partial y} = f_{yx} = -2e^{2x-y} + 6x^2y$$
$$\frac{\partial^2 f}{\partial z\partial y} = f_{yz} = 0$$
$$\frac{\partial^2 f}{\partial z^2} = f_{zz} == -\frac{1}{z^2}$$
$$\frac{\partial^2 f}{\partial x\partial z} = f_{zx} = 0$$
$$\frac{\partial^2 f}{\partial y\partial z} = f_{zy} = 0$$

Of course one can compute higher order derivatives. Note in the Example that $f_{xy} = f_{yx}$, $f_{xz} = f_{zx}$ and $f_{yz} = f_{zy}$.

**Theorem** (Clairaut 1713-1765) If $f$, $f_x$, $f_y$, $f_{xy}$ and $f_{yx}$ all exist and are continuous near $(a, b)$ then

$$f_{xy}(a, b) = f_{yx}(a, b)$$

The order of differentiation does not matter! In the previous example, we need only have computed 6 and then we know all 9 second partials. For functions of two variables we need compute only 3 second partials to know all 4.

**Definition** A function $f(x, y)$ is differentiable at $(x_0, y_0)$ if there exist constants $A$ and $B$ so that

$$f(x, y) = f(x_0, y_0) + A(x - x_0) + B(y - y_0) + (x - x_0)\epsilon_1 + (y - y_0)\epsilon_2$$

and

$$\lim_{(x,y)\to(x_0,y_0)} \epsilon_1 = 0 = \lim_{(x,y)\to(x_0,y_0)} \epsilon_2$$

Physically the definition says that the graph $z = f(x,y)$ has a tangent plane ($z = A(x-x_0) + B(y-y_0)$) at $(x_0, y_0, f(x_0, y_0))$

**Theorem** If $f$ is differentiable at $(x_0, y_0)$ then the partial derivatives exist at $(x_0, y_0)$ and $A = f_x(x_0, y_0)$ and $B = f_y(x_0, y_0)$

$$f(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) + (x-x_0)\epsilon_1 + (y-y_0)\epsilon_2$$

so that the tangent plane to the graph of $f$ is $z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$.

If $f_x$ and $f_y$ exist and are continuous on an open set then $f$ is differentialble on that open set but it is not enough to have $f_x$ and $f_y$ exist at a point. Example: See the transparency.

**Example** Find an equation for the tangent plane to the surface $z = x^+y^3$ at (1,1).

**Solution** $z_x = 2x = 2$ at (1,1) and $z_y = 3y^2 = 3$at (1,1) and so the tangent plane is $z = 2 + 2(x-1) + 3(y-1)$.

**Theorem** If $f(x,y)$ is differentiable at $(x_0, y_0)$ then it is continuous there.

**Proof** As in the single variable case.