14.4 The Chain Rule: Consider a composite function $f \circ \vec{r}(t)$ where $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$ and f is a differentiable function of two variables $f(x, y)$. Then

$$\lim_{h\to 0} \frac{1}{h} \left[f(x(t_0+h), y(t_0+h)) - f(x(t_0), y(t_0)) \right]$$
$$= \lim_{h\to 0} \frac{1}{h} [f_x(x(t_0), y_0(t_0))(x(t_0+h) - x(t_0)) + f_y(x(t_0), y_0(t_0))(y(t_0+h) - y_0(t_0))$$
$$+ (x(t_0+h) - x(t_0))\epsilon_1 + (y(t_0+h) - y(t_0))\epsilon_2]$$
$$= f_x(x(t_0), y_0(t_0)) \lim_{h\to 0} \frac{1}{h}(x(t_0+h) - x(t_0)) + f_y(x(t_0), y_0(t_0)) \lim_{h\to 0} \frac{1}{h}(y(t_0+h) - y(t_0))$$
$$+ \lim_{h\to 0} \frac{1}{h}(x(t_0+h) - x(t_0))\epsilon_1 + \lim_{h\to 0} \frac{1}{h}(y(t_0+h) - y(t_0))\epsilon_2$$
$$= f_x(x(t_0), y_0(t_0))x'(t_0) + f_y(x(t_0), y_0(t_0))y'(t_0) + 0$$

because ϵ_1 and ϵ_2 go to 0 as $x(t_0+h)$ goes to $x(t_0)$ and $y(t_0+h)$ goes to $y(t_0)$

Chain Rule for $f(x(t), y(t))$

$$\frac{d}{dt} f(x(t), y(t)) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Example Use the chain rule to find dw/dt if $w = x^2 y$ along the hyerbola $x = \sec t$ and $y = tant$

Solution

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$
$$= 2xy \sec t \tan t + x^2 (\sec t)^2 = 2 \sec t \tan t \sec t + (\sec t)^2 (\sec t)^2 = (\sec t)^2 [2 \tan t + (\sec t)^2]$$

There are other versions of the chain rule.

Chain Rule for $f(x(t), y(t), z(t))$

$$\frac{d}{dt} f(x(t), y(t), z(t)) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

Chain Rule for $f(x(r, s), y(r, s), z(r, s))$

$$\frac{\partial}{\partial r} f(x(r,s), y(r,s), z(r,s)) = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial r}$$
$$\frac{\partial}{\partial r} f(x(r,s), y(r,s), z(r,s)) = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial r}$$

This rule follows from the previous because the partial derivative is computed by regarding the other variable (s or r) as a constant.

Example Express $\partial f/\partial r$ and $\partial f/\partial \theta$ in terms of f_x and f_y for $f(x, y)$ where $x = r\cos\theta$ and $y = r\sin\theta$. Apply your formula to $f = x^2 y - xy^2$

Solution

$$\begin{aligned}\frac{\partial f}{\partial r} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} \\ &= \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta \\ \frac{\partial f}{\partial \theta} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} \\ &= \frac{\partial f}{\partial x}(-r\sin\theta) + \frac{\partial f}{\partial y}(r\cos\theta)\end{aligned}$$

and in the case $f = x^2y - xy^2$, $f_x = 2xy - y^2 = 2r^2\cos\theta\sin\theta - r^2(\sin\theta)^2$ and $f_y = x^2 - 2xy = r^2(\cos\theta)^2 - 2r^2(\sin\theta)(\cos\theta)$

Branch Diagrams Suppose $f(x, y, z)$ is a function depending on three variables x, y and z. These variables, themselves depend on two variables r and s. Then to determine the chain rule to find f_r and f_s we can draw the following branch diagram. so that

$$f_r = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial r}$$

and similarly

$$f_s = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$$

Implicit Differentiation Revisited Recall, from Calculus I, that an equation $F(x, y) = 0$ defines y as a function of x which may be difficult to solve for explicitly. (Here F is a differentiable function of two variables.) If we differentiate in x we get

$$F_x\frac{dx}{dx} + F_y\frac{dy}{dx} = 0$$

so that

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$