14.5 Directional Derivatives and the Gradient:
Definition The **gradient** of $f(x,y,z)$ is

$$\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$$

Example Find the gradient of $f(x,y) = y\cos xy + x^2y^3$.
Solution

$$\nabla f = (-y^2\sin xy + 2xy^3)\vec{i} + (\cos xy - xy\sin xy + 3x^2y^2)\vec{j}$$

Example Find the gradient of $f(x,y,z) = y^2e^{xz} + x^2z\ln y$
Solution

$$\nabla f = (y^2ze^{xz} + 2xz\ln y)\vec{i} + (2ye^{xz} + \frac{x^2z}{y})\vec{j} + (xy^2e^{xz} + x^2\ln y)\vec{k}$$

Directional Derivative Suppose that $\vec{u} = u_1\vec{i} + u_2\vec{j}$ is a *unit* vector $|\vec{u}| = 1$. The directional derivative of $f(x,y)$ in the direction u is

$$D_{\vec{u}}f = \frac{\partial f}{\partial x}u_1 + \frac{\partial f}{\partial y}u_2 = \nabla f \cdot \vec{u}$$

By the chain rule the function $h(t) = f((x+tu_1, y+tu_2))$ has derivative $h'(0) = D_{\vec{u}}f$ when $t = 0$. Picture:

Example Suppose a mountaineer is on the surface $z = 3x^2 - y^2$ at (1,2,-1) and she wishes to head northeast. What slope does she encounter?

Solution The slope is the directional derivative in the northeast direction that is in the direction of the vector $\vec{i} + \vec{j}$ that is the direction $\vec{u} = (1/\sqrt{2})[\vec{i} + \vec{j}]$ The slope is

$$D_{\vec{u}}z = \nabla f \cdot \vec{u} = \left[\frac{\partial z}{\partial x}\vec{i} + \frac{\partial z}{\partial y}\vec{j}\right] \cdot \left[\frac{1}{\sqrt{2}}\vec{i} + \frac{1}{\sqrt{2}}\vec{j}\right] = [6\vec{i} - 4\vec{j}] \cdot \left[\frac{1}{\sqrt{2}}\vec{i} + \frac{1}{\sqrt{2}}\vec{j}\right] = \frac{6}{\sqrt{2}} - \frac{4}{\sqrt{2}} = \sqrt{2}$$

She encounters a slope of $\sqrt{2}$ (about 0.955 radians or 55 degrees above horizontal.)

Example In the previous Example it is reasonable to ask what is the greatest slope our intrepid mountaineer would encounter at (1,2,-1)? What is the maximum of

$$D_{\vec{u}}z = \nabla f \cdot \vec{u} = |\nabla f||\vec{u}|\cos\theta = |\nabla f|\cos\theta$$

over all choices of the unit vector $\vec{u}$. We want $\theta = 0$ to maximize $\cos\theta$ and so $\vec{u}$ has to point in the same direction as ∇f:

$$\vec{u} = \frac{\nabla f}{|\nabla f|} = \frac{6\vec{i} - 4\vec{j}}{\sqrt{52}}$$

Therefore the gradient points in the direction of most rapid increase of f and that rate of increase is

$$D_{\vec{u}}f = |\nabla f|| = \sqrt{52}$$

Similarly

$$-\frac{\nabla f}{|\nabla f|} = -\frac{6\vec{i} - 4\vec{j}}{\sqrt{52}}$$

is the direction of most rapid decreased of f ($\theta = \pi$) and the rate of decrease is $-|\nabla f\| = -\sqrt{52}$

Physical Interpretation I of ∇f. The direction $\dfrac{\nabla f}{|\nabla f|}$ is the direction of most rapid increase of f and that rate of increase is $|\nabla f|$.

The opposite direction ($-\dfrac{\nabla f}{|\nabla f|}$) is the direction of most rapid decrease of f and that rate of decrease is $-|\nabla f|$.

Application: Water flow on the surface $z = f(x, y)$. Water flows in the direction $-\nabla f$.

Level Curves and the Gradient. Consider the level curve $f(x, y) = c$ of f. It is possible under most circumstances (assuming f is differentiable) to parameterize the curve $\vec{r} = x(t)\vec{i} + y(t)\vec{j}$. (This is a version of the Implicit Function Theorem of Advanced Calculus.) Therefore $f(x(t), y(t)) = f \circ \vec{r}(t) = c$ is constant. Differentiate both sides.

$$\nabla f \cdot \vec{r}\,' = \frac{\partial f}{\partial x} x' + \frac{\partial f}{\partial y} y' = 0$$

by the chain rule. This shows that ∇f is orthogonal to $\vec{r}\,' = x'\vec{i} + y'\vec{j}$, that is orthogonal to the tangent to the curve. We say ∇f is orthogonal to the level curve.

Physical Interpretation II of ∇f. The direction $\dfrac{\nabla f}{|\nabla f|}$ at any point is perpendicular to the level curve (for $f(x, y)$) or level surface (for $f(x, y, z)$) through that point.

Example Consider $f(x, y, z) = 4x^2 + y^2 + 9z^2$ at the point $(1, 2, -1)$. The level surface of f that passes through (1,2,-1) is $f = 17$

$$4x^2 + y^2 + 9z^2 = 17$$

We have $\nabla f = 4x\vec{i} + 2y\vec{j} + 18z\vec{k}$ and at (1,2,-1), $\nabla f = 4\vec{i} + 4\vec{j} - 18\vec{k}$ and this vector is perpendicular to the level surface at (1,2,-1) and points in the direction of most rapid increase of f. The directional derivative of f in the direction $(1/\sqrt{340})(4\vec{i} + 4\vec{j} - 18\vec{k}) = (1/\sqrt{85})(2\vec{i} + 2\vec{j} - 9\vec{k}$ is $\sqrt{340} = 2\sqrt{85}$