14.6 Tangent Planes and Differentials:

Example Find the tangent plane to the surface $x^3+xy^2+2y^3+z^2=4$ at (-1,1,-2). Find also the normal line to the surface at (-1,1,-2).

Solution This surface is the level surface of the function $f(x,y,z)=x^3+xy^2+2y^3+z^2$. Therefore we have a normal to the surface:

$$\nabla f=(3x^2+y^2)\vec{i}+(2xy+6y^2)\vec{j}+2z\vec{k}\text{so that}\nabla f(-1,1,2)=4\vec{i}+4\vec{j}-4\vec{k}$$

Therefore an equation for the tangent plane is

$$4(x+1)+4(y-1)-4(z+2)=0 \text{ or } x+y-z=2$$

The normal line is

$$\vec{r}=(-1+4t)\vec{i}+(1+4t)\vec{j}+(-2-4t)\vec{k}$$

(that is $x=-1+4t$, $y=1+4t$ and $z=-2-4t$.)

Example Find the gradient of $f(x,y,z)=y^2e^{xz}+x^2z\ln y$

The surface $f(x,y,z)=c$ (c is a constant) has **tangent plane** at $P_0=(x_0,y_0,z_0)$ (on the surface, so that $f(x_0,y_0,z_0)=c$)

$$f_x(P_0)(x-x_0)+f_y(P_0)(y-y_0)+f_z(P_0)(z-z_0)=0$$

The **normal line** is

$$x=x_0+f_x(P_0)t,\quad y=y_0+f_y(P_0)t,\quad z=z_0+f_z(P_0)t$$

Example Find the tangent plane and the normal line to the graph $z=\tan xy+x^2\ln y$ at $(\pi/4,1)$.

We see that the graph of $F(x,y)=\tan xy+x\ln y$ is the level surface of $f(x,y,z)=\tan xy+x\ln y-z$ at $(\pi/4,1,1)$ and so we can apply the formula above. We have $f_x=F_x=(\sec xy)^2y+\ln y$, $f_y=F_y=(\sec xy)^2x+x/y$ and $f_z=-1$. At $(\pi/4,1,1)$, $f_x=2$, $f_y=3\pi/4$ so that the tangent plane is

$$2(x-\frac{\pi}{4})+\frac{3\pi}{4}(y-1)-(z-1)=0 \text{ or } z=1+2(x-\frac{\pi}{4})+\frac{3\pi}{4}(y-1)$$

The normal line is $x=\pi/4+2t$, $y=1+(3\pi/4)t$ and $z=1-t$.

The surface $z=F(x,y)$ has the **tangent plane** at $P_0=(x_0,y_0,F(x_0,y_0))$

$$z=F(x_0,y_0)+F_x(x_0,y_0)(x-x_0)+F_y(x_0,y_0)(y-y_0)$$

The **normal line** is

$$x=x_0+F_x(x_0,y_0)t,\quad y=y_0+F_y(x_0,y_0)t,\quad z=F(x_0,y_0)-t$$

Example Find parametric equations for the tangent line to the curve of intersection of the two surfaces

$$4x^2-y^2+9z^2=36$$
$$-2x+5y+z=8$$

at(3,3,-1).

Solution: The tangent line is tangent to both surfaces and so it is perpendicular to both normals. The normal to the hyperboloid of one sheet is $\nabla f = (8x)\vec{i} - 2y\vec{j} + 18z\vec{k}$ and so at (3,3,-1) it is $\nabla f = (24)\vec{i} - 6\vec{j} - 18\vec{k} = 6(4\vec{i} - \vec{j} - 3\vec{k})$. The normal to the plane is, of course $\vec{N} = -2\vec{i} + 5\vec{j} + \vec{k}$. These vectors are perpendicular to the curve of intersection and so their cross product is parallel to the curve.

$$\begin{aligned}\nabla f \times \vec{N} &= 6\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & -1 & -3 \\ -2 & 5 & 1 \end{bmatrix} \\ &= 6(14\vec{i} + 2\vec{j} + 18\vec{k}) = 12(7\vec{i} + \vec{j} + 9\vec{k})\end{aligned}$$

Consequently the tangent line to the curve of intersection is

$$\vec{r}(t) = (3 + 7t)\vec{i} + (3 + t)\vec{j} + (-1 + 9t)\vec{k}$$

or in the book's notation: $x = 3 + 7t, y = \ldots$.

Definition The linearization of $f(x, y)$ at (x_0, y_0), assuming f is differentiable at (x_0, y_0), is

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Error If f has continuous first and second partials and if $|f_{xx}| \leq M$, $|f_{xy}| \leq M$ and $|f_{yy}| \leq M$ for some constant $M > 0$ on some rectangle around (x_0, y_0) then

$$|f(x, y) - L(x, y)| \leq M(|x - x_0| + |y - y_0|)^2$$

Example If $f(x, y) = \sqrt{x^2 + y} + 2y$ then the linearization at (1,3) is: $f_x = x(x^2 + y)^{-1/2}$, $f_y = (1/2)(x^2 + y)^{-1/2} + 2$,

$$L(x, y) = 8 + (1/2)(x - 1) + (17/8)(y - 3)$$

Definition The **total differential** of $f(x, y)$ at (x_0, y_0), assuming f is differentiable at (x_0, y_0), is

$$df = f_x(x_0, y_0)dx + f_y(x_0, y_0)dy$$

There is a comparable formula in the case of functions of 3 variables

$$df = f_x(x_0, y_0, z_0)dx + f_y(x_0, y_0, z_0)dy + f_z(x_0, y_0, z_0)dz$$

Example The diagonal of an x by y rectangle is $\sqrt{x^2 + y^2}$. If the construction is such that the $x = 4 \pm 0.02$ and $y = 3 \pm 0.01$ then the diagonal is is $\sqrt{3^2 + 4^2} \pm df \approx 5 \pm df$ where

$$\begin{aligned}df =& f_x(x_0, y_0)dx + f_y(x_0, y_0)dy \\ =& x_0(x_0^2 + y_0^2)^{-1/2}(\pm 0.02) + y_0(x_0^2 + y_0^2)^{-1/2}(\pm 0.01) \sim \pm\left(\frac{4}{5}(0.02) + \frac{3}{5}(0.01)\right) = \pm 0.022\end{aligned}$$