14.7 Extreme Values and Saddle Points: As in Calculus I, we are interested in local maxima and minima. A function $f(x,y)$ has a **local maximum** (resp. **local minimum**) at (a,b) if $f(x,y) \leq f(a,b)$ (resp. $f(x,y) \geq f(a,b)$ for all (x,y) in a disk centered at (a,b).

A disk is the interior of a circle. The disk could be small or large.

Definition A function $f(x,y)$ has a critical point at (a,b) if f is not differentiable at (a,b) or $\nabla f(a,b) = \vec{0}$.

Theorem If f is defined on some set D and if f has a local max or min at (a,b) in D then either (a,b) is a critical point or (a,b) is a boundary point.

Recall that (a,b) is a **boundary point** of a set D if every open disk centered at (a,b) contains points inside D and outside D.

Example $f(x,y) = \sqrt{x^2+y^2}$ where D is a disk of radius $a > 0$ centered at the origin in the xy-plane. Then f has exactly one critical point at the (0,0) because the partial derivatives there do not exist. ($f(x,0) = |x|$.) This corresponds to a local minimum. If D contains a boundary point then that is a local maximum. If D has no boundary points $D = \{(x,y) : x^2+y^2 < a^2\}$ then there are no local max and that is the same if D is the entire xy-plane.

Example $f(x,y) = y^2 - x^2$ has a critical point at (0,0) but it is neither a local max nor min: it is a **saddle point** because there are points (x,y) arbitrarily near (0,0) so that $f(x,y) > f(0,0)$ and other points so that $f(x,y) < f(0,0)$. In this example $f(0,y) > 0 = f(0,0)$ and $f(x,0) < 0$.

Closed Bounded Region Method This is an analogue of the closed interval method of Section 4.1.

Suppose $f(x,y)$ is continuous on a closed and bounded region R. (Thus R can be contained in a large enough ball and it contains all its boundary points.) Then f takes on both its absolute maximum value and its absolute minimum value in R and the points are either

1. critical point of f inside R or
2. at local extrema of f on the boundary point R

Then compare the values of f at all the points found and discover which is the absolute max and which is the absolute min.

Example Consider the function $f(x,y) = 2x^2 + y^2 - 2y$ on the triangle with vertices (0,0), (2,2) and (-2,2) assuming the edges and corners of the triangle are included.

Solution Sketch the region. This is a closed bounded region Check for critical points:

$\nabla f(x,y) = 4x\vec{i} + (2y-2)\vec{j}$. The critical points occur where $\nabla f(x,y) = \vec{0}$ or $\nabla f(x,y)$ does not exist.

$$\begin{aligned} 4x &= 0 \\ 2y - 2 &= 0 \end{aligned}$$

The only critical point is (0,1).

Check next the boundary. The boundary consists of the 3 edges and the 3 corners.

1. Edge from (0,0) to (2,2): Here $y = x$: $f(x, y) = f(x, x) = 2x^2 + x^2 - 2x = 3x^2 - 2x$, $0 \le x \le 2$
$$\frac{d}{dx}3x^2 - 2x = 6x - 2$$
So (1/3,1/3) is a critical point on the edge.
2. Edge from (0,0) to (-2,2): Here $y = -x$: $f(x, y) = f(x, -x) = 2x^2+x^2+2x = 3x^2+2x$, $-2 \le x \le 0$.
$$\frac{d}{dx}3x^2 + 2x = 6x + 2$$
So (-1/3,1/3) is a critical point on the edge.
3. Edge from (-2,2) to (2,2). Here $y = 2$: $f(x, 2) = 2x^2$ and
$$\frac{d}{dx}2x^2 = 4x$$
so that there is a critical point at (0,2).
4. The vertices (0,0), (2,2) and (-2,2).

Finally we have isolated all possible points where f could take on its absolute max and min values and we need only evaluate.

Point P	f(P)
$(0, 1)$	f(0,1) =-1
$(1/3, 1/3)$	f(1/3,1/3)=-1/3
$(-1/3, 1/3)$	f(-1/3,1/3)=-1/3
$(0, 2)$	f(0,2)=0
$(0, 0)$	f(0,0)=0
$(2, 2)$	f(2,2)= 8
$(-2, 2)$	f(-2,2)=8

The absolute maximum value 8 occurs at (2,2) and (-2,2) and the absolute minimum value -1 occurs at (0,1).

If the region is not closed and bounded then there is no foolproof method for finding absolute extrema but we do have a criterion for the local extrema.

Recall the:

The Second Derivative Test for functions $f(x)$ of a single variable. If $f'(a) = 0$ then $f''(a) > 0$ implies f has a local minimum at a and $f''(a) < 0$ implies f has a local max at a and $f''(a) = 0$ or $f''(a)$ does not exist then this test is indeterminant. (We don't have a clue.)

The Second Derivative Test for Functions $f(x,y)$ If $\nabla f(a,b) = \vec{0}$ and if the second partials exist near (a,b) and if $D = f_{xx}(a,b)f_{yy}(a,b) - (f_{xy}(a,b))^2$ (the **Discriminant**) then

1. if $D > 0$ and $f_{xx}(a,b) < 0$ then f has a local maximum at (a,b).
2. if $D > 0$ and $f_{xx}(a,b) > 0$ then f has a local minimum at (a,b).
3. if $D < 0$ then f has a saddle at (a,b).
4. if $D = 0$ or if the second partials do not exist near (a,b) then the test is "indeterminant" (does not work).

Example

1. $f(x,y) = x^2 + y^2$ has a local minimum at (0,0)
2. $f(x,y) = x^3 + y^3$ has a local saddle at (0,0) but the test is indeterminant ($D = 0$).
3. $f(x,y) = x^4 + y^4$ has a local minimum at (0,0) but the test is indeterminant.

Example: Find all the local maxima, minima and saddle points of $f(x,y) = x^3 + 12xy + 8y^3$

Solution: Find all critical points $\nabla f = (3x^2 + 12y)\vec{i} + (12x + 24y^2)\vec{j}$. F is differentiable everywhere and so the only critical points are where $\nabla f = \vec{0}$ We solve the (non linear!) system

$$3x^2 + 12y = 012x + 24y^2 = \qquad 0$$

from which we see $x^2 = -4y$ and $x = -2y^2$. Square both sides of the first equation $x^2 = 4y^4$ and combined with the first equation implies $4y^4 = -4y$ or $y(y^3 + 1) = 0$. Therefore $y = 0$ or $y = -1$. Substituting to find x we have (0,0) and (-2,-1) are the two critical points. (We check that they satisfy the equation $\nabla f = \vec{0}$.) Next we classify the critical points using the second derivative test. Compute the second partial derivatives: $f_{xx} = 6x$, $f_{yy} = 48y$ and $f_{xy} = f_{yx} = 12$ and so $D = 288xy - 144$. We only need the value of D at the critical points.

1. **(0,0)**: Here $D = -144 < 0$ and so (0,0) is a saddle point.
2. **(-2,-1)**: Here $D > 0$ and $f_{xx}(-2,-1) = -12 < 0$ and so (-2,-1) is a local maximum.

There is a similar test for functions $f(x,y,z)$ of three variables.