Multiple Integrals: Our goal is to use integration to find, for example, the mass of a three dimensional solid of known density (mass per volume). If the density is 1 then we find the volume. We did this in Chapter 6 for bodies with rotational symmetry or, at least, known cross section. Here we make minimal assumptions on the solid

15.1 Double Integrals over Rectangles: We begin by trying to find the volume of a solid bounded above by a surface $z = f(x, y)$ and below by a rectangle R in the xy–plane: $R: \; a \leq x \leq b, \; c \leq y \leq d$ Therefore $f(x, y) \geq 0$ on R for this to make physical sense. We subdivide R into small rectangles as in the picture

If the rectangles are small enough in area $\Delta A = \Delta x \Delta y$ then the volume under the graph of f but above the small rectangle is approximately

$$f(x, y)\Delta A$$

provided (x, y) is somewhere in the rectangle ΔA and f is bounded. The volume under $z = f(x, y)$ and above R is approximately

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i, y_j)\Delta A_{(i,j)} = \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i, y_j)\Delta x_i \Delta y_j$$

where the little rectangles $\Delta A_{(i,j)}$ subdivide all of R and are Δx_i by Δy_j and (x_i, y_j) belongs to $\Delta A_{(i,j)}$. We say that $f(x, y)$ is Riemann integrable over R if these approximations tend to a limiting value as the small rectangles $A_{(i,j)}$ get smaller and more numerous and their "diameter" gets smaller. The volume is that limiting value:

$$\iint_R f(x, y)\, dA$$

(or in some texts $\displaystyle\iint_R f(x, y)\, dx\, dy$).

It can be shown that, provided f is continuous or, if it has discontinuities then they are on "small" sets (like lines or curves) the f is Riemann integrable.

Example: Find the volume of the solid bounded above $f(x, y) = 2x^2 + y^2$ above the rectangle $R = \{(x, y) : 1 \leq x \leq 2, 0 \leq y \leq 2\}$.

Solution: The volume is $V = \iint_R f(x, y)\, dA$ but what is that? Let's find the area of a cross section of the solid cut by a plane $x = x_i$ where $1 \leq x_i \leq 2$. The area is

$$\int_0^2 f(x_i, y)\, dy = \int_0^2 2x_i^2 + y^2\, dy = \left[2x_i^2 y + \frac{1}{3}y^3\right]_0^2 = 4x_i^2 + \frac{8}{3}$$

The cross sectional area is $4x^2 + \frac{8}{3}$, $1 \le x \le 2$ (dropping the x_i notation). Now recall from Chapter 6, the formula for volume V,

$$V = \int_a^b A(x)\,dx$$

where $A(x)$ is the cross sectional area and the solid lie entirely between the planes $x = a$ and $x = b$. Here $A(x) = 4x^2 + (8/3)$ so that the volume is

$$V = \int_1^2 4x^2 + (8/3)\,dx = \frac{4}{3}x^3 + \frac{8}{3}x|_1^2 = \frac{4}{3}[2^3] + \frac{16}{3} - (\frac{4}{3} + \frac{8}{3} = 12$$

Solution 2: Let us interchange the roles of x and y to see what we get in that case. The area of a cross section of the solid perpendicular to the y–axis is

$$\int_1^2 f(x,y)\,dx(= A(y)) = \int_1^2 2x^2 + y^2\,dx = \frac{2}{3}x^3 + xy^2|_1^2 = \frac{16}{3} - \frac{2}{3} + 2y^2 - y^2 = \frac{14}{3} + y^2$$

and again we can get the volume as $V = \int_0^2 A(y)\,dy$:

$$V = \int_0^2 \frac{14}{3} + y^2\,dy = \frac{14}{3}y + \frac{1}{3}y^3|_0^2 = \frac{28}{3} + \frac{8}{3} - 0 = 12$$

Fubini's Theorem (First Form) If $f(x,y)$ is Riemann integrable on the rectangle $R = [a,b] \times [c,d]$ then

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

Remark The theorem says that the double integral $\iint_R f(x,y)\,dA$ can be evaluated using "iterated integrals" and either order.