15. Double Integrals over General Regions: We begin by trying to find the volume of a solid bounded above by a surface $z = f(x, y)$ and below by a region R in the xy–plane and R need not be a rectangle. As before the volume is given by

$$\iint_R f(x,y)\,dA$$

How do we integrate over a general region like R?

Let us suppose that our region R can be expressed as the region between the graph of two functions

Type I $R = \{(x, y) : g_1(x) \le y \le g_2(x), a \le x \le b\}$

R is bounded above by the graph $y = g_2(x)$ and below by $y = g_1(x)$. PICTURE!

Type II $R = \{(x, y) : h_1(y) \le x \le g_2(y), c \le y \le d\}$

R is bounded on the right by the graph $y = g_2(x)$ and on the left by $y = g_1(x)$. PICTURE!

Example: Find the volume of the solid bounded above by the graph of $f(x, y) = 2x^2y$ and below by the region in the xy plane $R = \{(x, y) : x^2 \le y \le 4, 0 \le x \le 2\}$.

Solution: The volume is $V = \iint_R f(x, y)\,dA$. How do we evaluate this? We look for the area $A(x)$ of the cross section corresponding to a fixed values of x, $0 \le x \le 2$. Sketch R.

$$A(x) = \int_{x^2}^{4} 2x^2y\,dy = x^2y^2|_{x^2}^{4} = 16x^2 - x^6$$

The volume of the solid is therefore $V = \int_0^2 A(x)\,dx$ or

$$V = \int_0^2 \int_{x^2}^{4} 2x^2y\,dy\,dx = \int_0^2 16x^2 - x^6\,dx = \frac{16}{3}x^3 - \frac{1}{7}x^7|_0^2 = \frac{128}{3} - \frac{128}{7} = \frac{512}{21}$$

Solution 2 The region R can also be described as a type II domain: $R = \{(x, y) : 0 \le x \le \sqrt{y}, 0 \le y \le 4\}$ so that

$$\begin{aligned} V = \int_0^4 \int_0^{\sqrt{y}} 2x^2y\,dx\,dy &= \int_0^4 \frac{2}{3}x^3y|_0^{\sqrt{y}}\,dy \\ &= \frac{2}{3}\int_0^4 y^{3/2}y - 0\,dy \\ &= \frac{2}{3}\frac{2}{7}y^{7/2}|_0^4 = \frac{512}{21} \end{aligned}$$

Fubini's Theorem (Second Form) If $f(x, y)$ is Riemann integrable on the rectangle R and

1. $R = \{(x, y) : a \le x \le b, g_1(x) \le y \le g_2(x)\}$ then

$$\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

2. OR $R = \{(x, y) : c \le y \le d, h_1(x) \le y \le h_2(x)\}$ then

$$\iint_R f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

Remark The theorem says that the double integral $\iint_R f(x,y)\,dA$ can be evaluated using "iterated integrals" and either order. The first case corresponds to R is type I and the second corresponds to R is type II.

Example Reverse the order of integration to evaluate the integral

$$\int_0^1 \int_y^1 x^2 \sin(xy)\,dx\,dy$$

Sketch the region of integration.

$$\int_0^1 \int_y^1 x^2 \sin(xy)\,dx\,dy = \int_0^1 \int_0^x x^2 \sin(xy)\,dy\,dx \tag{1}$$

$$= \int_0^1 [-x\cos(xy)]_0^x\,dx \tag{2}$$

$$= \int_0^1 [-x\cos(x^2) + x]\,dx = \int_0^1 -x\cos(x^2)\,dx + \frac{1}{2} \tag{3}$$

We can evaluate the integral by a u–substitution with $u = x^2$ and $du = 2x\,dx$

$$\int_0^1 -x\cos(x^2)\,dx + \frac{1}{2} = \frac{1}{2}[1 - \int_0^1 \cos u\,du] = \frac{1}{2}[1 - \sin 1]$$