15.4 Double Integrals in Polar Form: Certain doubles integrals $\iint_R f(x,y)\,dA$ are easier to evaluate if we take advantage of a point (the origin) symmetry. Possibly f is simpler in polar coordinates but more often the region is R is more easily specified in polar coordinates. The boundary of R might be a circle centered at the origin (r=constant) a circle through the origin ($r = a\cos\theta$ or $r = a\sin\theta$) or a cardioid ($r = a(1 \pm \cos\theta)$ or $r = a(1 \pm \sin\theta)$) or even a lemniscate ($r = a\sin 2\theta$)

Let's observe that we need to be a little careful here. For example a disk of radius $a > 0$ can be specified in polar coordinates by $0 \le r \le a$ and $0 \le \theta \le 2\pi$ But the area of a circle is πa^2 and NOT

$$\int_0^{2\pi}\int_0^a dr\,d\theta = 2\pi a$$

The integral does not know we are working polar coordinates and thinks we want the area of a rectangle that is $a \times 2\pi$. We have to modify our integration procedure so that it measure area correctly. We want to find the area of a region in the plane

$$\{(r,\theta) : r_0 \le r \le r_0 + \Delta r, \theta_0 \le \theta \le \theta_0 + \Delta\theta$$

(If we were in Cartesian coordinate then this would be a Δr by $\Delta\theta$ rectangle but in polar coordinates it is a segment of a annular ring. Picture)

The area of a sector of a circle of radius r whose opening angle is $\Delta\theta$ is $r^2\Delta\theta/2$ so that the area of the segment is

$$(r+\Delta r)^2\Delta\theta/2 - r^2\Delta\theta/2 = r\Delta r\Delta\theta + (\Delta r)^2\Delta\theta/2 \approx r\Delta r\Delta\theta$$

since Δr is very small.

This suggests that to find area of a region specified in polar coordinates we emply "$r\,dr\,d\theta$" instead of simply "$dr\,d\theta$" Check by using a double integral to find the area of a circle.

$$\int_0^{2\pi}\int_0^R r\,dr\,d\theta = \int_0^{2\pi} \frac{1}{2}r^2|_0^R, d\theta = [\frac{1}{2}R^2\theta]_0^{2\pi} = \pi R^2$$

Integration in Polar Coordinates:

$$\iint_R f(x,y)\,dA = \int_R f(r\cos\theta, \sin\theta)\,rdr\,d\theta$$

Example: Find the volume of the solid that is bounded above by the paraboloid $z = x^2 + y^2$ and lies above the region in the first quadrant $R = \{(x,y) : 1 \le x^2 + y^2 \le 4, 0 \le x, 0 \le y\}$.

Solution: The volume is $V = \iint_R x^2 + y^2\, dA$. This is a good candidate for polar coordinates because the R is simple in polar coordinates and so is $x^2 + y^2$. Therefore

$$V = \int_0^{\pi/2} \int_1^2 r^2 r\, dr\, d\theta = \int_0^{\pi/2} \frac{1}{4} r^4 |_1^2\, d\theta = \frac{15}{4}\frac{\pi}{2} = \frac{15\pi}{8}$$

Example: Find the volume of the solid region that is interior to both the sphere $x^2 + y^2 + z^2 = 4$ of radius 2 and the cylinder $(x-1)^2 + y^2 = 1$. This is the volume of material removed when an off-center hole of radius 1 is bored just tangent to a diameter all the way through the sphere.

Solution: The height function of this solid is $f(x, y) = 2\sqrt{4 - x^2 - y^2}$. The region R is the projection of the solid in the xy–plane which is the same as the interior of the cylinder and so bounded by $(x-1)^2 + y^2 = 1$. The volume is therefore $V = \iint_R 2\sqrt{4 - x^2 - y^2}\, dA$

In polar coordinates, the boundary of R is $(x-1)^2 + y^2 = 1$ or $x^2 + y^2 = 2x$ or $r^2 = 2r\cos\theta$ or $r = 2\cos\theta$, $-\pi/2 \leq \theta\pi/2$. Therefore

$$V = 2\int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} \sqrt{4 - r^2} r\, dr\, d\theta$$

We can evaluate the inner integral by a u–substitution: $u = 4 - r^2$, $du = -2r\, dr$,

$$\begin{aligned}
V = -\frac{2}{3}\int_{-\pi/2}^{\pi/2} u^{3/2}|_4^{4-4\cos^2\theta}\, du\, d\theta = \quad & -\frac{2}{3}\int_{-\pi/2}^{\pi/2} (4 - 4\cos^2\theta)^{3/2} - 4^{3/2}\, d\theta \\
= \frac{2}{3}\int_{-\pi/2}^{\pi/2} 8 - 8|\sin^3\theta|\, d\theta & \\
= \frac{16\pi}{3} - \frac{32}{3}\int_1^0 1 - u^2\, du & \\
= \frac{16\pi}{3} - \frac{64}{9} = \frac{16}{9}[3\pi - 4] &
\end{aligned}$$