15.5 Triple Integrals: Suppose that we have a solid D and we know the density (mass per unit volume) $F(x, y, z)$ at every point of D . We want to find the mass of D. We slice up D into tiny rectangular solids of volume ΔV where $F(x, y, z)$ is almost a constant $F(x_i, y_j, z_k)$ and the mass is therefore approximately $F(x_i, y_j, z_k)\Delta V$. The point (x_i, y_j, z_k) is inside the tiny rectangular solid. If we cut D into many tiny rectangular solids and add up those approximate masses we get an approximation of the mass of D as

$$\sum F(x_i, y_j, z_k)\Delta V$$

If D is a rectangular solid; $a \le x \le b$; $c \le y \le d$ and $e \le z \le f$ then we can subdivide the x–interval into segments of length Δx and similarly for the y and z–intervals so that $\Delta V = (\Delta x)(\Delta y)(\Delta z)$ then the above sum is

$$\sum_{i=1}^{\ell}\sum_{j=1}^{m}\sum_{k=1}^{n} F(x_i, y_j, z_k)(\Delta x)(\Delta y)(\Delta z)$$

where it is understood that (x_1, y_1, z_1) is in the "first" small rectangular block and (x_1, y_1, z_2) is in the block above that and so on.

In the case that D is a more general solid but we still know the density $F(x, y, z)$ then we can still use a triple sum like above but we must omit terms if the small rectangular block does not touch D but include all terms where the rectangular block is entirely inside D.

Now we look at what happens when the small blocks get smaller and the number consequently gets larger. Then our estimate gets more accurate and if we approach a limiting value no matter how we choose the blocks (but the size of the blocks must go to zero) then we say that F is Riemann integrable on D and we denote the limiting value by

$$\iiint_D F(x, y, z)\, dV = \iiint_D F(x, y, z)\, dx\, dy\, dz$$

and that is indeed the mass M of D: $M = \iiint_D F(x, y, z)\, dV$.

How do we evaluate a triple integral? We might slice D horizontally into layers that are Δz thick and find the mass of each layer regarding z as a constant in each layer (Δz is very small). Each layer is roughly a region in a plane $z = z_k$ (z_k=constant) and the mass should be $\iint_{D\cap\{z=z_k\}} F(x, y, z_k)\, dx\, dy \Delta z$ roughly. We can evaluate this double integral as in previous Sections. To get the total mass we then add up the mass of each layer using integration in z. If D is between the planes $z = a$ and $z = b$ then

$$\iiint_D F(x,y,z)\,dV = \int_a^b \iint_{D(z)} F(x,y,z)\,dA\,dz$$

where $D(z) = \{(x,y) : (x,y,z) \in D\}$. Of course, we have chosen to slice parallel to the xy–plane but we could equally well have held x constant (parallel to the yz–plane)

$$\iiint_D F(x,y,z)\,dV = \int_a^b \iint_{D(x)} F(x,y,z)\,dA\,dx$$

or y constant

$$\iiint_D F(x,y,z)\,dV = \int_a^b \iint_{D(y)} F(x,y,z)\,dA\,dy$$

Theorem (Fubini) If $F(x,y,z)$ is Riemann integrable on D then the three integration methods all give $\iiint_D F\,dV$

Example Find the mass of the rectangular solid D bounded by the planes $x = 1$, $x = 2$, $y = 1$, $y = 3$, $z = 1$ and $z = 5$ if density at (x,y,z) is $F(x,y,z) = 2x + 5y + 7z$.

Mass is

$$M = \iiint_D F\,dV = \int_1^5 \int_R (2x + 5y + 7z)\,dA\,dz = \int_1^5 \int_1^3 \int_1^2 2x + 5y + 7z\,dx\,dy\,dz$$

where R denotes the rectangle $1 \le x \le 2$, $1 \le y \le 3$. We can evaluate

$$\begin{aligned} M &= \int_1^5 \int_1^3 x^2 + 5xy + 7xz|_1^2\,dy\,dz \\ &= \int_1^5 \int_1^3 3 + 5y + 7z\,dy\,dz \\ &= \int_1^5 3y + \frac{5}{2}y^2 + 7zy|_1^3\,dz \\ &= \int_1^5 6 + 20 + 14z\,dz \\ &= \left[26z + 7z^2\right]_1^5 = 272 \end{aligned}$$

Example Find the mass of a tetrahedron bounded by the plane $x + 2y + 2z = 4$ and the three coordinate planes and lying in the first octant. Assume constant density.

Solution Sketch the solid. It is bounded above by the plane $x + 2y + 2z = 4$ which intersects the coordinate axes at (4,0,0), (0,2,0) and (0,0,2).

The mass of the solid D is M where

$$\begin{aligned}
M &= \iiint_D \delta \, dV \\
&= \int_0^4 \int_0^{2-x/2} \int_0^{2-y-x/2} \delta \, dz \, dy \, dx \\
&= \delta \int_0^4 \int_0^{2-x/2} z|_0^{2-y-x/2} \, dy \, dx \\
&= \delta \int_0^4 \int_0^{2-x/2} 2 - y - x/2 \, dy \, dx \\
&= \delta \int_0^4 2y - \frac{1}{2}y^2 - xy/2|_0^{2-x/2} \, dx \\
&= \delta \int_0^4 2(2-x/2) - \frac{1}{2}(2-x/2)^2 - \frac{x}{2}(2-x/2) \, dx \\
&= \delta \int_0^4 2 - x + x^2/8 \, dx \\
&= \delta 2x - x^2/2 + x^3/24|_0^4 = \frac{8}{3}\delta
\end{aligned}$$

The first moment of inertia M_{yz} can be computed similarly. (Multiply the integrands by x until evaluating the x-integral.)

$$\begin{aligned}
M &= \iiint_D \delta x \, dV \\
&= \delta \int_0^4 2x - x^2 + x^3/8 \, dx \\
&= \delta x^2 - \frac{1}{3}x^3 + x^4/32|_0^4 = \frac{8}{3}\delta
\end{aligned}$$

We shall see in the next section $\overline{x} = M_{yz}/M$ is the xth coordinate of the center of mass (balance point). In the previous example $\overline{x} = 1$. Therefore if D were placed on a knife edge one unit from the base (in the yz–plane) then it would balance.

Example: Express the mass of the region D which is bounded by the planes $x = y$, $z = 0$ and $y = 0$ and the cylinder $x^2 + z^2 = 1$ and has density $\delta = z$ as 6 different triple integrals using Fubini's theorem

Solution: The mass is (see the photocopy)

$$M = \int$$