Consider a solid D in $\mathbb{R}^3$ with density $\delta = \delta(x, y, z)$ (in units of mass per unit volume). Then the mass is

$$M = \iiint_D \delta \, dV$$

The first moments are

About the x–axis

$$M_{yz} = \iiint_D x\delta \, dV$$

About the y–axis

$$M_{xz} = \iiint_D y\delta \, dV$$

About the z–axis

$$M_{xy} = \iiint_D z\delta \, dV$$

and the center of mass is

$$(\overline{x}, \overline{y}, \overline{z}) = \left(\frac{M_{yz}}{M}, \frac{M_{xz}}{M}, \frac{M_{xy}}{M}\right)$$

The second moments (or moments of inertia) are

About the x–axis

$$I_x = \iiint_D (y^2 + z^2)\delta \, dV$$

About the y–axis

$$I_y = \iiint_D (x^2 + z^2)\delta \, dV$$

About the z–axis

$$I_z = \iiint_D (x^2 + y^2)\delta \, dV$$

For example if a body is rotating about the x axis at ω radians per second then the kinetic energy of rotation is $(1/2)w^2 I_x$

Example: Find the center of mass of the block that lies below the plane $z = 3 + x + y$ and above the square R: $-1 \leq x \leq 1$, $-1 \leq y \leq 1$ in the xy–plane assuming constant density δ.

Solution: The mass is

$$\begin{aligned} M = \iiint_D \delta \, dV &= \int_{-1}^{1} \int_{-1}^{1} \int_{0}^{3+x+y} \delta \, dz \, dy \, dx \\ &= \delta \int_{-1}^{1} \int_{-1}^{1} 3 + x + y \, dy \, dx \\ &= \delta \int_{-1}^{1} 3y + xy + \frac{1}{2} y^2 |_{-1}^{1} \, dx \\ &= \delta \int_{-1}^{1} 6 + 2x + 0 \, dx \\ &= \delta 6x + x^2 |_{-1}^{1} = 12\delta \end{aligned}$$

Note that all terms involving an odd power of x and/or y integrate to 0. The first moment about the x–axis is

$$
\begin{aligned}
M_{yz} = \iiint_D \delta x \, dV = & \int_{-1}^{1} \int_{-1}^{1} \int_{0}^{3+x+y} x\delta \, dz \, dy \, dx \\
=& \delta \int_{-1}^{1} \int_{-1}^{1} 3x + x^2 + xy \, dy \, dx \\
=& \delta \int_{-1}^{1} 3xy - x^2 y + \frac{1}{2} x y^2 |_{-1}^{1} \, dx \\
=& \delta \int_{-1}^{1} 6x + 2x^2 - 2x - 0 \, dx \\
=& \delta 3x^2 + \frac{2}{3} x^3 |_{-1}^{1} = \frac{4}{3}\delta
\end{aligned}
$$

and so $\overline{x} = M_{yz}/M = 1/9$ We see by symmetry (interchange x and y) $M_{xz} = 4/3$ and $\overline{y} = M_{xz}/M = 1/9$. It remains to calculate

$$
\begin{aligned}
M_{xy} = \iiint_D \delta z \, dV = & \int_{-1}^{1} \int_{-1}^{1} \int_{0}^{3+x+y} z\delta \, dz \, dy \, dx \\
=& \frac{\delta}{2} \int_{-1}^{1} \int_{-1}^{1} (3 + x + y)^2 \, dy \, dx \\
=& \frac{\delta}{2} \int_{-1}^{1} \int_{-1}^{1} 9 + 6x + 6y + x^2 + y^2 + 2xy \, dy \, dx \\
=& \frac{\delta}{2} \int_{0}^{2} 9y + x^2 y + \frac{1}{3} y^3 |_{-1}^{1} \, dx \\
=& \frac{\delta}{2} 18x + \frac{2}{3} x^3 + \frac{2}{3} x |_{x=-1}^{x=1} = (18 + \frac{4}{3})\delta = \frac{58\delta}{3}
\end{aligned}
$$

so that $\overline{z} = \frac{58}{3(12)} = \frac{29}{18}$

Example: Find the second moment of inertia of a circular cylinder of radius a about its axis of symmetry. Assume uniform density δ and (constant) height h.

Solution: Orient the axes so that the z–axis is the axis of symmetry and the xy–plane bisects the cylinder. The cylinder is all (x, y) so that $x^2 + y^2 \leq a^2$. The second moment of inertia about the z axis is

$$
\begin{aligned}
\iiint_D (x^2 + y^2)\delta \, dV = & \int_{-h/2}^{h/2} \int_R x^2 + y^2 \delta \, dA \\
=& \delta \int_{-h/2}^{h/2} \int_{0}^{2\pi} \int_{0}^{a} r^2 r \, dr \, d\theta \\
=& \delta \frac{h}{4} a^4 (2\pi) = \frac{\pi}{2} a^4 h\delta
\end{aligned}
$$

Notice energy growing as a^4.

Example: Find the second moment of inertia about the z–axis of the solid D if D is the region in the first quadrant bounded above by the surface $z = x^2 + 2y^2$ and bounded by the three coordinate planes and the plane $2x + y = 2$. Assume D has constant density. Picture

$$
\begin{aligned}
\iiint_D (x^2+y^2)\delta\, dV &= \iint_R \int_0^{x^2+2y^2} (x^2+y^2)\delta\, dz\, dA \\
&= \iint_R (x^2+2y^2)(x^2+y^2)\delta\, dA \\
&= \int_0^1 \int_0^{2-2x} (x^4 + 3x^2y^2 + 2y^4)\delta\, dy\, dx \\
&= \int_0^1 \left[x^4 y + x^2 y^3 + \frac{2}{5} y^5 \right]_0^{2-2x} \delta\, dx \\
&= \int_0^1 \left[x^4(2-2x) + x^2(2-2x)^3 + \frac{2}{5}(2-2x)^5 \delta \right] dx \\
&= \int_0^1 \left[2x^4 - 2x^5 + 8x^2 - 24x^3 + 24x^4 - 8x^5 + \frac{64}{5}(1-x)^5 \right] dx = \quad \delta \left[\frac{2}{5} - \frac{1}{3} + \frac{8}{3} - 6 + \right.
\end{aligned}
$$