15.7 Triple Integrals in Cylindrical and Spherical Coordinates

Example: Find the second moment of inertia of a circular cylinder of radius a about its axis of symmetry. Assume uniform density δ and (constant) height h.

Solution: Orient the axes so that the z–axis is the axis of symmetry and the xy–plane bisects the cylinder. The cylinder is all (x, y) so that $x^2 + y^2 \le a^2$, $-h/2 \le z \le h/2$. The second moment of inertia about the z axis is

$$\begin{aligned}\iiint_D (x^2+y^2)\delta\, dV =& \int_{-h/2}^{h/2} \int_R x^2 + y^2 \delta\, dA\, dz \\ =&\delta \int_{-h/2}^{h/2} \int_0^{2\pi} \int_0^a r^2 r\, dr\, d\theta \\ =&\delta \frac{h}{4} a^4 (2\pi) = \frac{\pi}{2} a^4 h\delta\end{aligned}$$

Notice energy growing as a^4.

Note the use of "cylindrical coordinates" (r, θ, z) and the use of $r\, dr\, d\theta\, dz$ or $dz\, r\, dr\, d\theta$.

Example: Set up but do not evaluate an integral in cylindrical coordinates that represents the mass of the portion of the solid bounded by $z = 4$ and $z^2 = x^2 + y^2$ that lies in the first octant if the density is $\delta = x^2 y$

Solution This is one quarter (or one eighth) of a cone. Draw a picture

$$\begin{aligned}\iiint_D x^2 y\, dV =& \int_0^{\pi/2} \int_0^4 \int_r^4 r^2 (\cos\theta)^2 r \sin\theta\, dz r\, dr\, d\theta \\ =& \int_0^{\pi/2} \int_0^4 r^4 [z]_r^4 (\cos\theta)^2 \sin\theta\, dr\, d\theta \\ =& \int_0^{\pi/2} \int_0^4 [4r^4 - r^5] (\cos\theta)^2 \sin\theta\, dr\, d\theta \\ =& \int_0^{\pi/2} \frac{4}{5} r^5 - \frac{1}{6} r^6 |_0^4 (\cos\theta)^2 \sin\theta\, d\theta \\ =&4^6 [\frac{1}{5} - \frac{1}{6}] \int_0^{\pi/2} (\cos\theta)^2 \sin\theta\, d\theta \\ =&\frac{4^6}{30} \left[-\frac{1}{3} (\cos\theta)^3 \right]_0^{\pi/2} = \frac{2^{11}}{45}\end{aligned}$$

Spherical Co-ordinates: The points in three space can be specified by three real numbers: ρ is the distance to the origin ($\rho = \sqrt{x^2 + y^2 + z^2}$; $\rho \ge 0$) and ϕ is the angle made with the positive z axis ($0 \le \phi \le \pi$) and θ is as in cylindrical coordinates: the angle made with respect to the positive x–axis. $0 \le \theta \le 2\pi$. The conversions are

$$\begin{aligned} z =& \rho \cos\phi \\ x =& \rho \sin\phi \cos\theta \\ y =& \rho \sin\phi \sin\theta \\ r =& \rho \sin\phi \\ \rho =& \sqrt{x^2 + y^2 + z^2} \end{aligned}$$

ExamplesIdentify and /or sketch.

1. $\rho = 11$
2. $\theta = \pi/4$
3. $\phi = \pi/4$
4. $\rho = 3\cos\phi$

Volumes in Spherical Coordinates If there is a symmetry about the origin in 3–space in either the solid D and/or the integrand $f(x, y, z)$ then one considers spherical coordinates but if we just integrate in ρ, ϕ and θ then our integral is designed for Cartesion coordinates and volumes. If we consider the solid in three space specified by $\rho_i \le \rho \le \rho_i + \Delta\rho$, $\theta_j \le \theta \le \theta_j + \Delta\theta$ $\phi_k \le \phi \le \phi_k + \Delta\phi$. See the picture. The volume is roughly $\rho_i^2 \sin\phi_k \Delta\rho\Delta\theta\Delta\phi$.

Converting integrals from Cartesian to Spherical.

$$\iiint_D f(x, y, z)\, dV = \iiint_D f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \cos\phi)\rho^2 \sin\phi d\rho d\theta d\phi$$

On teh left side of the equation D must be described in Cartesian coordinates; on the right it should be described in spherical coordinates.

Example Find the volume of the solid that lies above the cone $z = \sqrt{3}\sqrt{x^2 + y^2}$ but below the sphere $x^2 + y^2 + z^2 = 2z$.

Solution These two surfaces are specified easily in spherical coordinates. The cone is $\rho\cos\phi = \sqrt{3}\rho\sin\phi$ or $\tan\phi = 1/\sqrt{3}$ or $\phi = \pi/6$. The sphere is $\rho^2 = 2\rho\cos\phi$ or $\rho = 2\cos\phi$

$0 \leq \phi \leq \pi/2$. The volume is

$$\begin{aligned}
V &= \int_0^{\pi/6} \int_0^{2\pi} \int_0^{2\cos\phi} \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi \\
&= \int_0^{\pi/6} \int_0^{2\pi} \left[\frac{1}{3}\rho^3\right]_0^{2\cos\phi} \sin\phi \, d\theta \, d\phi \\
&= \frac{8}{3} \int_0^{\pi/6} \int_0^{2\pi} [\cos\phi]^3 \sin\phi \, d\theta \, d\phi \\
&= \frac{16\pi}{3} \int_0^{\pi/6} [\cos\phi]^3 \sin\phi \, d\phi \\
&= \frac{16\pi}{3} \left[-\frac{1}{4}(\cos\phi)^4\right]_0^{\pi/6} \\
&= \frac{4\pi}{3}(1 - (\cos\pi/6)^4) = \frac{7\pi}{12}
\end{aligned}$$