16.1 and 16.2 Vector Fields and Line Integrals

A vector field is a function from $\mathbb{R}^3$ to itself.

$$\vec{F}(x,y,z) = M(x,y,z)\vec{i} + N(x,y,z)\vec{j} + P(x,y,z)\vec{k}$$

To each point in space $\vec{F}$ assigns a vector.

Applications: (a) Velocity Fields At each point on earth the wind is blowing with a certain velocity. That velocity field is the function that assigns to every point on the earth or every point in the atmosphere the wind velocity. See page 927.

(b) Force Fields. At every point near the surface of the earth gravity exerts a force towards the earth's center. The force on a mass m is

$$\vec{F}(x,y,z) = -\frac{GMm}{(x^2+y^2+z^2)^{3/2}}\left[x\vec{i} + y\vec{j} + z\vec{k}\right]$$

where G is the gravitational constant and M is the mass of the earth and (x,y,z) is near but "above the surface of the earth

Abstract Line Integrals: Recall that the length of a curve C parameterized by $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$, $a \le t \le b$ is

$$\int_C ds = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\,dt$$

An abstract line integral of f along C is

$$\int_C f\,ds = \int_a^b f(x(t),y(t),z(t))\sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\,dt$$

It might represent the mass of a wire which has variable density $f(x(t),y(t),z(t))$ in units of mass per unit length.

Example Suppose that a particle is traveling through a liquid in the path

$$\vec{r}(t) = 2\cos t\vec{i} + 2\sin t\vec{j} + t\vec{k}, \qquad 0 \le t \le \pi$$

and the liquid is of variable density $f(x,y,z) = x + 2y + 3z$ (in grams per cubic centimeter) and suppose that the particle causes a thin wire to solidify along its path. The wire is of constant cross sectional area A. What is the mass of the wire?

$$\begin{aligned}\int_C f\,ds =& A\int_0^\pi 2\cos t + 4\sin t + 3t\sqrt{(-2\sin t)^2 + (2\cos t)^2 + 1}\,dt\\ =& \sqrt{5}A\int_0^\pi 2\cos t + 4\sin t + 3t\,dt = \sqrt{5}A\left[2\sin t - 4\cos t + \frac{3}{2}t^2\right]_0^\pi = \frac{3}{2}A\pi^2\end{aligned}$$

16.2 Work If a constant force $\vec{F}$ acts on an object through a displacement of $\vec{u}$ then the work done by $\vec{F}$ is $\vec{F}\cdot\vec{u} = |\vec{F}||\vec{u}|\cos\theta$. Picture Observe that $\theta > \pi/2$ implies negative work is done. It is convenient to write work as

$$W = \vec{F} \cdot \vec{u} = \frac{\vec{F} \cdot \vec{u}}{|\vec{u}|}|\vec{u}| = \mathrm{comp}_{\vec{u}}(\vec{F})|\vec{u}|$$

Work is the component of $\vec{F}$ along $\vec{u}$ times the distance displaces.

Now consider the case the $\vec{F}$ is not constant and acts along a curve $\vec{r}$, $a \leq t \leq b$. How much work is done? Over a short enough time period $\vec{F}$ is essentially constant and so the work done over the time period t_i to $t_i + \Delta t$ is approximately

$$\vec{F}(t_i) \cdot \vec{r}'(t_i)\Delta t$$

because $\vec{r}'(t_i)\Delta t$ is approximately the displacement. If we add up over choices of i (so that the intervals $[t_i, t_i + \Delta]$ partition $[a, b]$ we get total work done is approximately

$$\begin{aligned} \sum_{i=1}^{n} \vec{F}(t_i) \cdot \vec{r}'(t_i)\Delta t &\approx \int_a^b \vec{F}(t_i) \cdot \vec{r}'(t_i)\, dt \\ &= \int_a^b \frac{\vec{F}(t_i) \cdot \vec{r}'(t_i)}{|\vec{r}'(t)|}|\vec{r}'(t)|\, dt \\ &= \int_C \frac{\vec{F}(t_i) \cdot \vec{r}'(t_i)}{|\vec{r}'(t)|}\, ds \end{aligned}$$

Work is therefore a line integral $\int_C f\, ds$ where f is the component of the force $\vec{F}(t)$ along the unit tangent vector $\vec{T}(t) = \vec{r}'(t_i)|\vec{r}'(t)|$ Work done by $\vec{F}$ along C is

$$\begin{aligned} \int_C \vec{F} \cdot d\vec{r} &= \int_C \vec{F} \cdot \vec{T}(t) ds \\ &= \int_C M\, dx + N\, dy \\ &= \int_a^b M(x(t), y(t), z(t))x'(t) + N(x(t), y(t), z(t))y'(t)\, dt \end{aligned}$$

where C is parameterized by $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$, $a \leq t \leq b$.

Example Find the work done by $\vec{F}(x, y, z) = x^2\vec{i} - xy\vec{j}$ along the curve C which consists of a quarter circle or radius 1 centered at the origin from (1,0) and then (0,1) and followed by the vertical straight line segment from (0,1,) to (0,2). Picture

Solution: We should parameterize the curve.

$$\begin{aligned} \vec{r}_1(t) &= \cos t\vec{i} + \sin t\vec{j}, \quad 0 \leq t \leq \pi/2 \\ \vec{r}_2(t) &= (1 + t)\vec{j}, \quad 0 \leq t \leq 1 \end{aligned}$$

The total work is therefore

$$\begin{aligned}
\int_C \vec{F}\cdot d\vec{r} = & \int_C M\,dx + N\,dy \\
= & \int_{C_1} M\,dx + N\,dy + \int_{C_2} M\,dx + N\,dy \\
= & \int_0^{\pi/2} (\cos t)^2(-\sin t) - (\cos t)(\sin t)\cos t + \int_0^1 t + 1\,dt \\
= & \frac{2}{3}(\cos t)^3|_0^{\pi/2} + \frac{1}{2}t^2 + t|_0^1 \\
= & -\frac{2}{3} + \frac{3}{2} = \frac{5}{6}
\end{aligned}$$

Flux. Here we suppose that the curve C is "closed" which means $\vec{r}(b) = \vec{r}(a)$ and we are interested in how much of the velocity flow $\vec{F}$ is outward across the curve. Flux is

$$\int_C \vec{F}\cdot\vec{n}ds = \int_C M dy - N dx$$

Here $\vec{n}$ denotes the unit outward normal. The tangent is $\vec{r}'(t) = x'(t)\vec{i} + y'(t)\vec{j}$ so that $y'(t)\vec{i} - x'(t)\vec{j}$ is outward and perpendicular to the tangent vector and so

$$\vec{n}(t) = \frac{y'(t)\vec{i} - x'(t)\vec{j}}{\sqrt{x'(t)^2 + y'(t)^2}}$$

so that

$$\begin{aligned}
\int_C \vec{F}\cdot\vec{n}ds = & \int_a^b (M\vec{i} + N\vec{j})\cdot\frac{y'(t)\vec{i} - x'(t)\vec{j}}{\sqrt{x'(t)^2 + y'(t)^2}}\sqrt{x'(t)^2 + y'(t)^2}\,dt \\
= & \int_a^b M\,y'(t) - N\,x'(t)\,dt = \int_C M dy - N dx
\end{aligned}$$

Example Find the flux of $\vec{F}(x, y, z) = x^2\vec{i} + (y + xy)\vec{j}$ outward through the curve C: $x^2/9 + y^2/4 = 1$.

Parameterize the curve $x = 3\cos t$,$y = 2\sin t$, $0 \le t \le 2\pi$ Flux is

$$\begin{aligned}
\int_C M\,dy - N\,dx = & \int_0^{2\pi} (3\cos t)^2 2\cos t - [2\sin t + (3\cos t)(2\sin t)](-3\sin t)\,dt \\
= & \int_0^{2\pi} (18\cos t)^3 + 18\cos t(\sin t)^2 + 6(\sin t)^2\,dt \\
= & \int_0^{2\pi} 18\cos t + 3(1 - \cos 2t)\,dt = 6\pi
\end{aligned}$$

Work?