8.2 Trigonometric Integrals:

Identities:

1. $(\sin x)^2 + (\cos x)^2 = 1$
2. $(\cos x)^2 = \frac{1}{2}(1 + \cos 2x)$
3. $(\sin x)^2 = \frac{1}{2}(1 - \cos 2x)$
4. $(\sec x)^2 = (\tan x)^2 + 1$
5. $(\csc x)^2 = (\cot x)^2 + 1$
6. $\sin 2x = 2 \sin x \cos x$
7. $\cos 2x = (\cos x)^2 - (\sin x)^2$

Integrating: $\int (\sin x)^m (\cos x)^n \, dx$

There are 3 cases

1. n **is odd**: Substitute $u = \sin x$, $du = \cos x \, dx$. **2.** m **is odd**: Substitute $u = \cos x$, $du = -\sin x \, dx$.

Example: Evaluate $\int (\sin x)^3 (\cos x)^3 \, dx$.

Solution: Let $u = \sin x$, $du = \cos x \, dx$. This leaves $(\cos x)^2 = 1 - (\sin x)^2 = 1 - u^2$.

$$\int (\sin x)^3 (\cos x)^3 \, dx = \int u^3(1 - u^2) \, du = u^4/4 - u^6/6 = \frac{1}{4}(\sin x)^4 - \frac{1}{6}(\sin x)^6 + C$$

The third case is:

3. m **and** n **are both even:** Substitute $(\cos x)^2 = \frac{1}{2}(1 + \cos 2x)$ and $(\sin x)^2 = \frac{1}{2}(1 - \cos 2x)$

Example: Evaluate $\int (\cos x)^4 (\sin x)^2 \, dx$.

Solution: Here

$$(\cos x)^4 (\sin x)^2 = \left(\frac{1}{2}(1 + \cos 2x)\right)^2 \frac{1}{2}(1 - \cos 2x) = \frac{1}{8}(1 + \cos 2x - (\cos 2x)^2 - (\cos 2x)^3)$$

Therefore

$$\begin{aligned} \int (\cos x)^4 (\sin x)^2 \, dx &= \frac{1}{8} \int 1 + \cos 2x - (\cos 2x)^2 - (\cos 2x)^3 \, dx \\ &= \frac{1}{8}\left(x + \frac{1}{2}\sin 2x - \int (\cos 2x)^2 \, dx - \int (\cos 2x)^3 \, dx\right) \end{aligned}$$

Apply the identity $(\cos 2x)^2 = (1 + \cos 4x)/2$ to the first integral above.

$$\int (\cos 2x)^2 \, dx = \frac{1}{2} \int (1 + \cos 4x) \, dx = x/2 + (\sin 4x)/8 + C$$

As for the second integral let $u = \sin 2x$, $du = 2\cos 2x$ and use the identity $(\cos 2x)^2 = 1 - (\sin 2x)^2 = 1 - u^2$

$$\int (\cos 2x)^3\,dx = \frac{1}{2}\int 1 - u^2\,du = \frac{1}{2}(u - u^3/3) = (\sin 2x)/2 - (\sin 2x)^3/6$$

so that finally

$$\begin{aligned}\int (\cos x)^4(\sin x)^2\,dx &= x/8 + (\sin 2x)/16 - x/16 - (\sin 4x)/64 - (\sin 2x)/16 \\ &\quad + (\sin 2x)^3/48 + C \\ &= x/16 - (\sin 4x)/64 + (\sin 2x)^3/48 + C\end{aligned}$$

(This checks.)

Integrating: $\int (\tan x)^m (\sec x)^n\,dx$

There are 3 cases:
1. n **is even**: Substitute $u = \tan x$.
2. m **is odd**: Substitute $u = \sec x$
3. neither: Here $\int (\sec x)^n\,dx$ and n is odd. Case by case.
Example: Evaluate $\int (\sec x)^4\,dx$
Solution: $u = \tan x$, $du = (\sec x)^2\,dx$ and $(\sec x)^2 = (\tan x)^2 + 1 = u^2 + 1$.

$$\int (\sec x)^4\,dx = \int u^2 + 1\,du = u^3/3 + u + C = (\tan x)^3/3 + tanx + C$$

Check by differentiation

$$\frac{d}{dx}[(\tan x)^3/3 + tanx] = (\tan x)^2(\sec x)^2 + (\sec x)^2 = (\sec x)^4.$$

Example: Evaluate $\int (\tan x)^3\,dx$
Solution: $u = \sec x$ so that $du = \sec x \tan x\,dx$ which means $du/u = \tan x\,dx$. Write $(\tan x)^2 = (\sec x)^2 - 1 = u^2 - 1$. Therefore

$$\int (\tan x)^3\,dx = \int (u^2 - 1)\frac{1}{u}\,du = \int u - \frac{1}{u}\,du = u^2/2 - \ln|u| + C = (\sec x)^2/2 + \ln|\cos x| + C$$

Example: Evaluate $\int \sec x\,dx$.
Solution: Memorize this one.

$$\int \sec x\,dx = \int \sec x \frac{secx + \tan x}{\sec x + \tan x}\,dx = \int \frac{(secx)^2 + \sec x \tan x}{\sec x + \tan x}\,dx$$

Now let $u = \sec x + \tan x$ so that $du = ((secx)^2 + \sec x \tan x)\,dx$ Therefore

$$\int \sec x\,dx = \int \frac{1}{u}\,du = \ln|\sec x + \tan x| + C$$

For $\int (\sec x)^3\,dx$ see the text p 447.

Identities:

1. $\sin A \cos B = \frac{1}{2}[\sin(A - B) + \sin(A + B)]$
2. $\sin A \sin B = \frac{1}{2}[\cos(A - B) - \cos(A + B)]$
3. $\cos A \cos B = \frac{1}{2}[\cos(A - B) + \cos(A + B)]$