# LESSON 1 RADIAN ANGLE MEASUREMENT

Topics in this lesson:



## 1. MAKING ANGLES

How to make an angle: First, start with a ray (  ). Place another ray on top of the first. Then rotate the second ray either clockwise or counterclockwise.

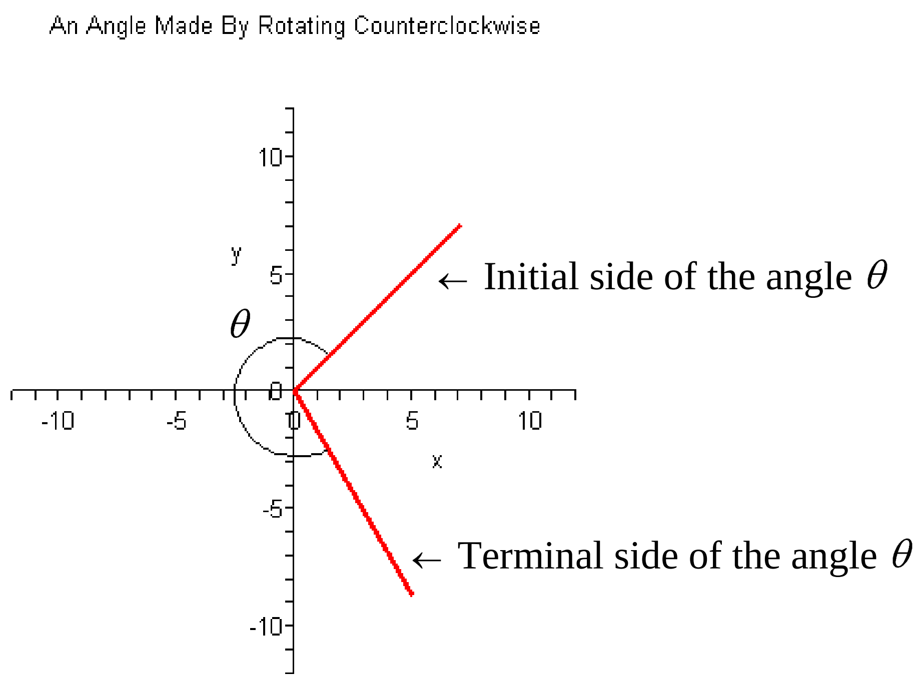
**Example** Animation of an angle being made by rotating counterclockwise.

**Example** Animation of an angle being made by rotating clockwise.

The measurement of an angle is positive when the angle is made by rotating counterclockwise and the measurement of an angle is negative when the angle is made by rotating clockwise.

**NOTE:** One complete revolution counterclockwise makes the angle $360°$. Two complete revolutions counterclockwise makes the angle $720°$. Five complete revolutions counterclockwise makes the angle $1800°$. Ten complete revolutions clockwise makes the angle $-3600°$. Thus, there is no largest angle and there is no smallest angle.

**Terminology:** The first ray that you start with is the initial side of the angle. When you stop (or terminate) the rotating of the second ray, this second ray is the terminal side of the angle. The common point of both rays is the vertex of the angle.




**Definition** An angle is said to be in standard position if the initial side of the angle is the positive *x*-axis.

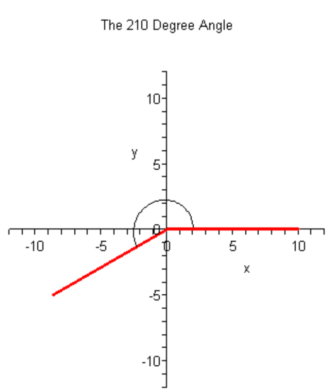
The initial side of the angle $\theta$ above is in the I (quadrant) and the terminal side of the angle is in the IV (quadrant). Thus, the angle $\theta$ is not in standard position.

**NOTE:** All the angles that we consider in this class will be in standard position.

When an angle is standard position, we know that the initial side of the angle is the positive *x*-axis. So, we only need to be told the location of the terminal side of the angle. The location of the terminal side of the angle could be I, II, III, IV, the positive *x*-axis, the negative *x*-axis, the positive *y*-axis, or the negative *y*-axis. Since we know that we are talking about the terminal side of the angle, it is redundant to say the terminal side. Thus, we will say the location of the angle instead of the location of the terminal side of the angle.

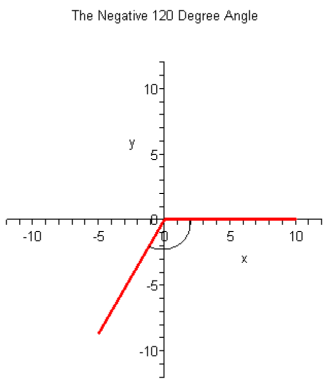
**Examples** Consider the making of the following angles (in standard position).

1. $\theta = 210°$



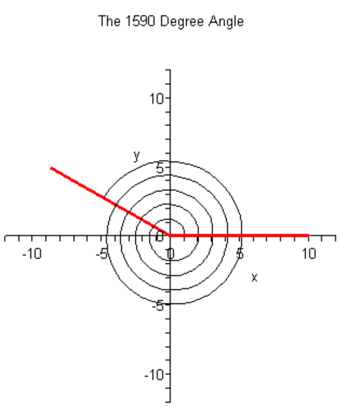

Animation of the making of this angle. Notice that the location of this angle is III.

2. $\alpha = -\ 120\,°$




Animation of the making of this angle. Notice that the location of this angle is III.

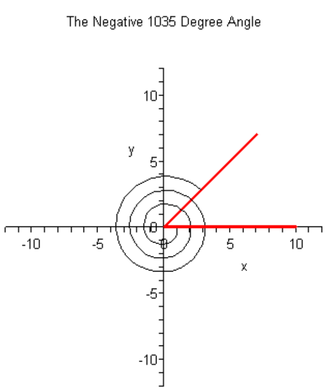
3. $\beta = 1590\,°$




Since each complete revolution is $360\,°$ and $360\,°$ divides into $1590\,°$ four times with a remainder of $150\,°$, then it will take four complete revolutions and an additional rotation of $150\,°$ going in the counterclockwise direction in order to make the $1590\,°$ angle.

Animation of the making of this angle. Notice that the location of this angle is II.

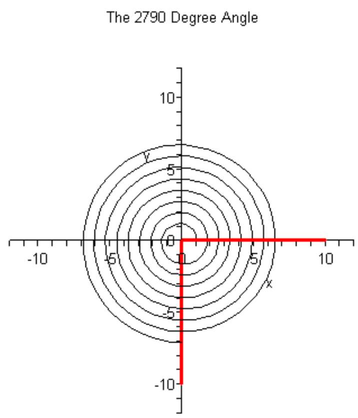
4. $\gamma = -\ 1035\,°$




Since $360\,°$ divides into $1035\,°$ two times with a remainder of $315\,°$, then it will take two complete revolutions and an additional rotation of $315\,°$ going in the clockwise direction in order to make the $-1035\,°$ angle.

Animation of the making of this angle. Notice that the location of this angle is I.

5. $\phi = 2790°$




Since $360°$ divides into $2790°$ seven times with a remainder of $270°$, then it will take seven complete revolutions and an additional rotation of $270°$ going in the counterclockwise direction in order to make the $2790°$ angle.

Animation of the making of this angle. Notice that the location of this angle is the negative *y*-axis.

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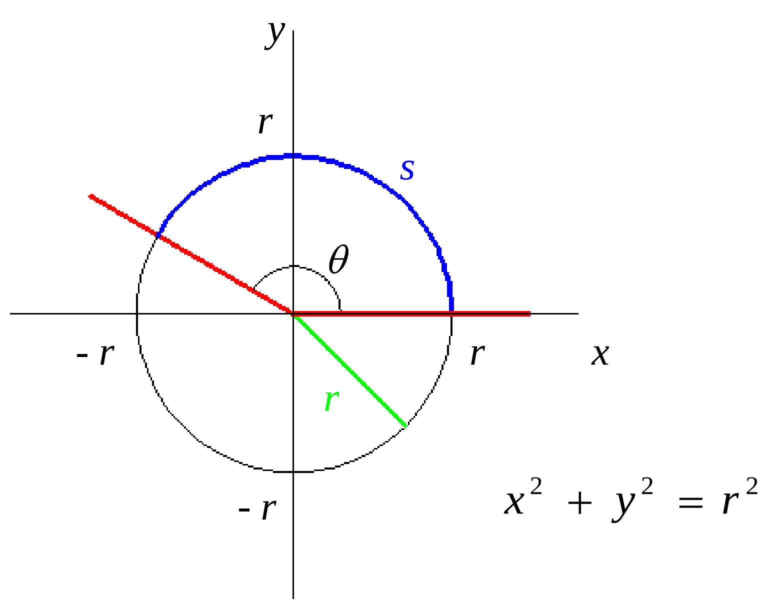
## 2. RADIAN ANGLE MEASUREMENT

Now, we will discuss Radian Angle Measurement. Consider the following animation. The animation shows that as the second ray is rotated, the point of intersection of the second ray and the circle of radius $r$ traces an arc of length $s$. Thus, $s$ is the measurement of the arc of the circle between the initial side of the angle (which is the positive *x*-axis) and the terminal side of the angle. These two numbers are used to determine the radian angle measurement of the angle $\theta$.



The radian angle measurement of $\theta$ is obtained by dividing the arc length *s* by the radius *r* of the circle. That is,

$$\theta = \frac{s}{r}$$

The angle $\theta$ is **positive** if the rotation is **counterclockwise**. The angle $\theta$ is **negative** if the rotation is **clockwise**. Since both *s* and *r* are positive numbers, you will have to supply this negative sign.

**NOTE:** Since you would use the same units (inches, feet, meters, etc.) to measure both *s* and *r*, then the radian measurement of an angle is dimensionless. A radian angle measurement is a real number.

Since you can use any size circle for radian angle measurement, the best circle to use would be the circle whose radius *r* is 1. The equation of this circle is $x^2 + y^2 = 1$. This circle is called the Unit Circle. When we use the Unit Circle, we get $\theta = s$. That is, radian angle measurement can be thought of as distance walked on the Unit Circle. That is, the radian angle measurement of the angle $\theta$ is the distance that you would walk on the Unit Circle from the initial side of the angle (which is the positive *x*-axis) to terminal side of the angle. If the angle is positive, you walk the distance going counterclockwise. If the angle is negative, you walk the distance going clockwise.

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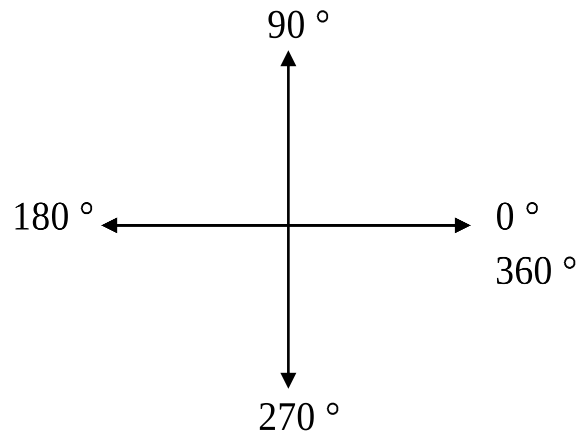
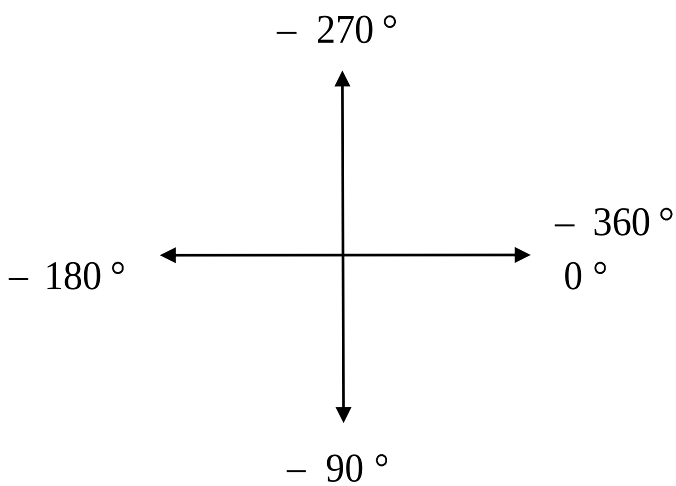
## 3. LOCATION OF ANGLES

Remember, all angles are considered to be in standard position. Thus, the initial side of the angle is the positive *x*-axis. So, when we talk about the location of an angle, we are talking about the location of the terminal side of the angle.

We know that we have the location of the following angles in degrees made going counterclockwise:




We know that we have the location of the following angles in degrees made going clockwise:



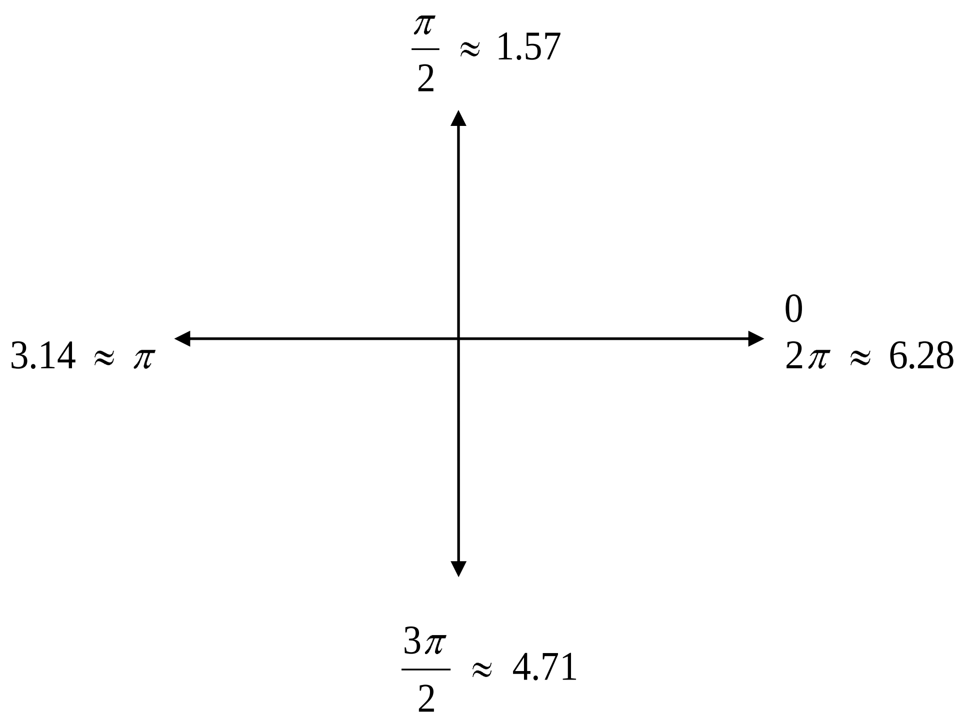

Since the circumference of the Unit Circle is $2\pi$, then if you walk one time around the Unit Circle either counterclockwise or clockwise, you would walk a distance of $2\pi$.

If you walk one-quarter of the Unit Circle either counterclockwise or clockwise, you would walk a distance of $\frac{\pi}{2}$.

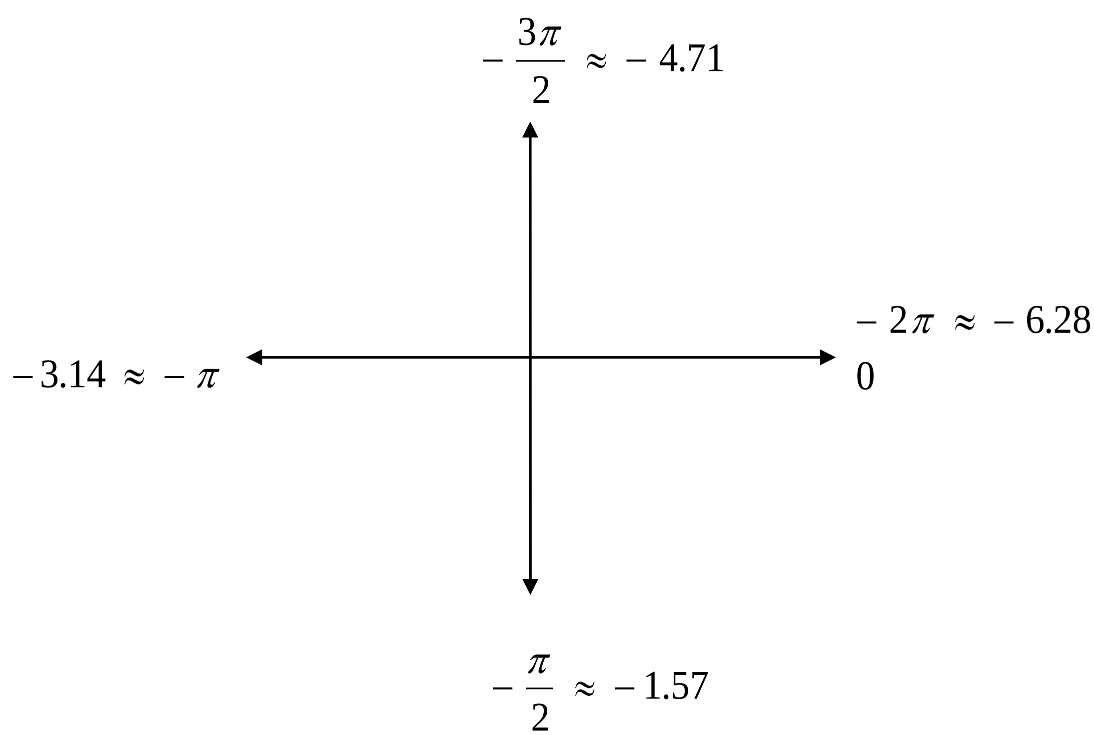
If you walk half of the Unit Circle either counterclockwise or clockwise, you would walk a distance of $\pi$.

If you walk three-quarters of the Unit Circle either counterclockwise or clockwise, you would walk a distance of $\frac{3\pi}{2}$.

Thus, we have the location of the following angles in radians made going counterclockwise:




Thus, we have the location of the following angles in radians made going clockwise:



**Examples** Determine the location of the following angles.

1. $\theta = 140°$

Since the angle is **positive**, then the angle is made by rotating **counterclockwise**.

First, compare the angle $140°$ with the angle $180°$. $140° < 180°$. Since the angle $\theta = 140°$ is being made by rotating **counterclockwise**, then the angle $\theta$ is in either I or II.

Now, compare the angle $140°$ with the angle $90°$. $140° > 90°$. Since the angle $\theta = 140°$ is being made by rotating **counterclockwise**, then the angle $\theta$ is in II.

Thus, the terminal side of this angle is in II.

**Answer:** II

2. $\alpha = -350°$

Since this angle is **negative**, then the angle is made by rotating **clockwise**. First, compare the angle $350°$ with the angle $180°$. $350° > 180°$. Since the angle $\alpha = -350°$ is being made by rotating **clockwise**, then the angle $\alpha$ is in either II or I.

Now, compare the angle $350°$ with the angle $270°$. $350° > 270°$. Since the angle $\alpha = -350°$ is being made by rotating **clockwise**, then the angle $\alpha$ is in I.

Thus, the terminal side of this angle is in I.

**Answer:** I

3. $\beta = \dfrac{9\pi}{7}$

Since the angle is **positive**, then angle is made by rotating **counterclockwise**. Since radian angle measurement can be thought of as distance walked on the Unit Circle, we want to know what quadrant, that we would be in, if we walk a distance of $\dfrac{9\pi}{7}$ counterclockwise on the Unit Circle starting at the point $(1, 0)$ on the positive *x*-axis.

First compare the distance of $\dfrac{9\pi}{7}$ with the distance of $\pi$ (walking half of the Unit Circle).

$\dfrac{9\pi}{7} > \pi = \dfrac{7\pi}{7}$ That is, the distance of $\dfrac{9\pi}{7}$ is **greater than** the distance of $\pi$. Thus, you have walked **more than half** of the Unit Circle. Since we are walking **counterclockwise** on the Unit Circle, then the angle $\beta = \dfrac{9\pi}{7}$ is in either III or IV.

Now, compare the distance of $\dfrac{9\pi}{7}$ with the distance of $\dfrac{3\pi}{2}$ (walking three-quarters of the Unit Circle).

$\dfrac{9\pi}{7} < \dfrac{3\pi}{2} = \dfrac{10.5\pi}{7}$ That is, the distance of $\dfrac{9\pi}{7}$ is **less than** the distance of $\dfrac{3\pi}{2}$. Thus, you have walked **less than three-quarters** of the Unit Circle.

Thus, we walked **counterclockwise** on the Unit Circle a distance greater than $\pi$ (half of the Unit Circle) **and** less than $\dfrac{3\pi}{2}$ (three-quarters of the Unit Circle). Thus, the terminal side of the angle $\beta = \dfrac{9\pi}{7}$ is in III.

Animation of the making of the angle $\beta = \dfrac{9\pi}{7}$.

**Answer:** III

4. $\phi = -\dfrac{9\pi}{7}$

Since the angle is **negative**, then angle is made by rotating **clockwise**. Since radian angle measurement can be thought of as distance walked on the Unit Circle, we want to know what quadrant, that we would be in, if we walk a distance of $\dfrac{9\pi}{7}$ clockwise on the Unit Circle starting at the point $(1, 0)$ on the positive *x*-axis.

First compare the distance of $\dfrac{9\pi}{7}$ with the distance of $\pi$ (walking half of the Unit Circle).

$\dfrac{9\pi}{7} > \pi = \dfrac{7\pi}{7}$ That is, the distance of $\dfrac{9\pi}{7}$ is **greater than** the distance of $\pi$. Thus, you have walked **more than half** of the Unit Circle. Since we are walking **clockwise** on the Unit Circle, then the angle $\phi = -\dfrac{9\pi}{7}$ is in either II or I.

Now, compare the distance of $\dfrac{9\pi}{7}$ with the distance of $\dfrac{3\pi}{2}$ (walking three-quarters of the Unit Circle).

$\dfrac{9\pi}{7} < \dfrac{3\pi}{2} = \dfrac{10.5\pi}{7}$ That is, the distance of $\dfrac{9\pi}{7}$ is **less than** the distance of $\dfrac{3\pi}{2}$. Thus, you have walked **less than three-quarters** of the Unit Circle.

Thus, we walked **clockwise** on the Unit Circle a distance greater than $\pi$ (half of the Unit Circle) **and** less than $\frac{3\pi}{2}$ (three-quarters of the Unit Circle). Thus, the terminal side of the angle $\phi = -\frac{9\pi}{7}$ is in II.

Animation of the making of the angle $\phi = -\frac{9\pi}{7}$.

**Answer:** II

5. $\gamma = -\frac{7\pi}{15}$

Since the angle is **negative**, then angle is made by rotating **clockwise**. Since radian angle measurement can be thought of as distance walked on the Unit Circle, we want to know what quadrant, that we would be in, if we walk a distance of $\frac{7\pi}{15}$ clockwise on the Unit Circle starting at the point $(1, 0)$ on the positive *x*-axis.

First compare the distance of $\frac{7\pi}{15}$ with the distance of $\pi$ (walking half of the Unit Circle).

$\frac{7\pi}{15} < \pi = \frac{15\pi}{15}$ That is, the distance of $\frac{7\pi}{15}$ is **less than** the distance of $\pi$. Thus, you have walked **less than half** of the Unit Circle. Since we are walking **clockwise** on the Unit Circle, then the angle $\gamma = -\frac{7\pi}{15}$ is in either IV or III.

Now, compare the distance of $\frac{7\pi}{15}$ with the distance of $\frac{\pi}{2}$ (walking one quarter of the Unit Circle).

$\frac{7\pi}{15} < \frac{\pi}{2} = \frac{7.5\pi}{15}$ That is, the distance of $\frac{7\pi}{15}$ is **less than** the distance of $\frac{\pi}{2}$. Thus, you have walked **less than one quarter** of the Unit Circle.

Thus, we walked **clockwise** on the Unit Circle a distance less than $\frac{\pi}{2}$ (one quarter of the Unit Circle). Thus, the terminal side of the angle $\gamma = -\frac{7\pi}{15}$ is in IV.

Animation of the making of the angle $\gamma = -\frac{7\pi}{15}$.

**Answer:** IV

6. $\theta = \frac{17\pi}{24}$

Since the angle is **positive**, then angle is made by rotating **counterclockwise**. Since radian angle measurement can be thought of as distance walked on the Unit Circle, we want to know what quadrant, that we would be in, if we walk a distance of $\frac{17\pi}{24}$ counterclockwise on the Unit Circle starting at the point $(1, 0)$ on the positive *x*-axis.

First compare the distance of $\frac{17\pi}{24}$ with the distance of $\pi$ (walking half of the Unit Circle).

$\frac{17\pi}{24} < \pi = \frac{24\pi}{24}$ That is, the distance of $\frac{17\pi}{24}$ is **less than** the distance of $\pi$. Thus, you have walked **less than half** of the Unit Circle. Since we are walking **counterclockwise** on the Unit Circle, then the angle of $\theta = \frac{17\pi}{24}$ is in either I or II.

Now, compare the distance of $\frac{17\pi}{24}$ with the distance of $\frac{\pi}{2}$ (walking one quarter of the Unit Circle).

$\frac{17\pi}{24} > \frac{\pi}{2} = \frac{12\pi}{24}$ That is, the distance of $\frac{17\pi}{24}$ is **greater than** the distance of $\frac{\pi}{2}$. Thus, you have walked **more than one quarter** of the Unit Circle.

Thus, we walked **counterclockwise** on the Unit Circle a distance less than $\pi$ (half of the Unit Circle) **and** greater than $\frac{\pi}{2}$ (one quarter of the Unit Circle). Thus, the terminal side of the angle $\theta = \frac{17\pi}{24}$ is in II.

Animation of the making of the angle $\theta = \frac{17\pi}{24}$.

**Answer:** II

7. $\alpha = -\frac{79\pi}{42}$

Since the angle is **negative**, then angle is made by rotating **clockwise**. Since radian angle measurement can be thought of as distance walked on the Unit Circle, we want to know what quadrant, that we would be in, if we walk a distance of $\frac{79\pi}{42}$ clockwise on the Unit Circle starting at the point $(1, 0)$ on the positive *x*-axis.

First compare the distance of $\frac{79\pi}{42}$ with the distance of $\pi$ (walking half of the Unit Circle).

$\frac{79\pi}{42} > \pi = \frac{42\pi}{42}$ That is, the distance of $\frac{79\pi}{42}$ is **greater than** the distance of $\pi$. Thus, you have walked **more than half** of the Unit Circle. Since we are walking **clockwise** on the Unit Circle, then the angle $\alpha = -\frac{79\pi}{42}$ is in either II or I.

Now, compare the distance of $\frac{79\pi}{42}$ with the distance of $\frac{3\pi}{2}$ (walking three-quarters of the Unit Circle).

$\frac{79\pi}{42} > \frac{3\pi}{2} = \frac{63\pi}{42}$ That is, the distance of $\frac{79\pi}{42}$ is **greater than** the distance of $\frac{3\pi}{2}$. Thus, you have walked **more than three-quarters** of the Unit Circle.

Thus, we walked **clockwise** on the Unit Circle a distance greater than $\frac{3\pi}{2}$ (three-quarters of the Unit Circle). Thus, the terminal side of the angle $\alpha = -\frac{79\pi}{42}$ is in I.

Animation of the making of the angle $\alpha = -\frac{79\pi}{42}$.

**Answer:** I

8. $\beta = 180\,°$

Since this angle is **positive**, then the angle is made by rotating **counterclockwise**. Thus, the terminal side of this angle is on the negative *x*-axis.

**Answer:** Negative *x*-axis

9. $\phi = -\dfrac{3\pi}{2}$

Since the angle is **negative**, then the angle is made by rotating **clockwise**. The distance of $\dfrac{3\pi}{2}$ is three-quarters of the Unit Circle. Thus, if we walk this distance clockwise, then the terminal side of the angle is on the positive *y*-axis.

Animation of the making of the angle $\phi = -\dfrac{3\pi}{2}$.

**Answer:** Positive *y*-axis

10. $\theta = 5$

Since the angle is **positive**, then the angle is made by rotating **counterclockwise**. Since radian angle measurement can be thought of as distance walked on the Unit Circle, we want to know what quadrant, that we would be in, if we walk a distance of 5 counterclockwise on the Unit Circle starting at the point $(1, 0)$ on the positive *x*-axis.

First compare the distance of 5 with the distance of $\pi$ (walking half of the Unit Circle).

$5 > \pi \approx 3.14$ That is, the distance of 5 is **greater than** the distance of $\pi$. Thus, you have walked **more than half** of the Unit Circle. Since we are walking **counterclockwise** on the Unit Circle, then the angle of $\theta = 5$ is in either III or IV.

Now, compare the distance of 5 with the distance of $\dfrac{3\pi}{2}$ (walking three-quarters of the Unit Circle).

$5 > \frac{3\pi}{2} \approx 4.71$ That is, the distance of 5 is **greater than** the distance of $\frac{3\pi}{2}$. Thus, you have walked **more than three-quarters** of the Unit Circle.

Thus, we walked **counterclockwise** on the Unit Circle a distance greater than $\frac{3\pi}{2}$ (three-quarters of the Unit Circle). Thus, the terminal side of the angle $\theta = 5$ is in IV.

Animation of the making of the angle $\theta = 5$.

**Answer:** IV

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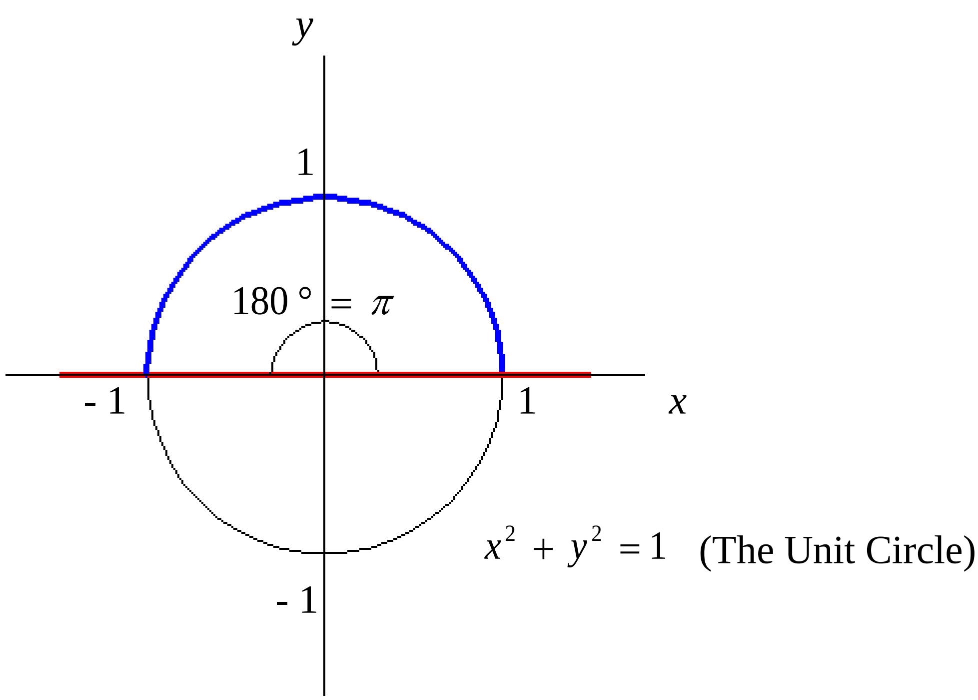
## 4. CONVERSION FACTORS FOR CONVERTING ANGLE UNITS




divide equation by $\pi$

$$180° = \pi \longrightarrow \frac{180°}{\pi} = 1$$

$\dfrac{180°}{\pi}$ is the conversion factor that converts from radians to degrees

$$180° = \pi \xrightarrow{\text{divide equation by } 180°} 1 = \frac{\pi}{180°}$$

$\dfrac{\pi}{180°}$ is the conversion factor that converts from degrees to radians

**<u>Examples</u>** Convert the following angles to radians if given in degrees or to degrees if given in radians.

1. $\theta = 325°$

$$\theta = 325° \cdot \frac{\pi}{180°} = 65 \cdot \frac{\pi}{36} = \frac{65\pi}{36}$$

**Answer:** $\dfrac{65\pi}{36}$

2. $\alpha = -\dfrac{11\pi}{6}$

$$\alpha = -\frac{11\pi}{6} \cdot \frac{180°}{\pi} = -\frac{11}{1} \cdot \frac{30°}{1} = -330°$$

**Answer:** $-330°$

3. $\beta = -240°$

$$\beta = -240° \cdot \frac{\pi}{180°} = -24 \cdot \frac{\pi}{18} = -4 \cdot \frac{\pi}{3} = -\frac{4\pi}{3}$$

**Answer:** $-\frac{4\pi}{3}$

4. $\gamma = \frac{7\pi}{15}$

$$\gamma = \frac{7\pi}{15} \cdot \frac{180°}{\pi} = \frac{7}{5} \cdot \frac{60°}{1} = \frac{7}{1} \cdot \frac{12°}{1} = 84°$$

**Answer:** $84°$

5. $\theta = -3$

$$\theta = -3 \cdot \frac{180°}{\pi} = -\frac{540°}{\pi} \quad \text{or} \quad \theta = -\frac{540}{\pi} \text{ degrees}$$

**Answer:** $-\frac{540°}{\pi}$

Animation of the making of the angle $\theta = -3$.

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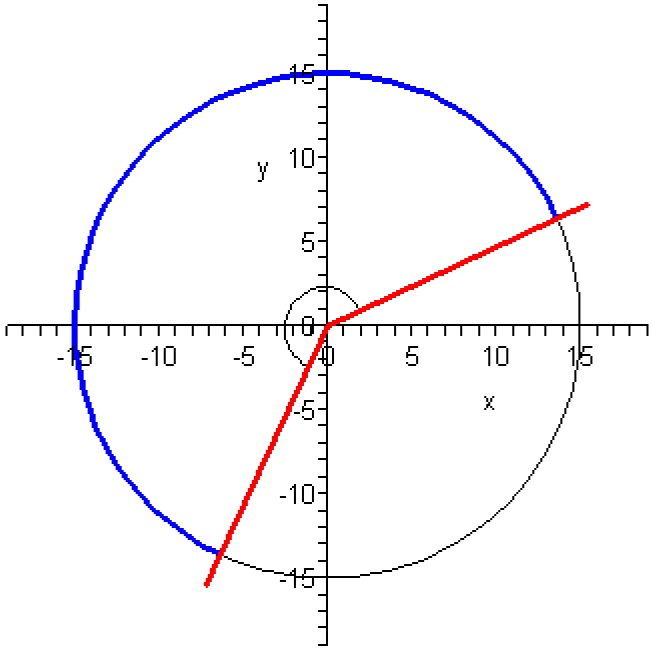
## 5. APPLICATION OF THE RADIAN ANGLE MEASUREMENT FORMULA

**Definition** If the vertex of an angle is the center of a circle, then the angle is called a central angle.

**Example** Find the length of the arc which is intercepted by a central angle of $220°$ on a circle of radius 15 feet.




Recall the Radian Angle Measurement Formula: $\theta = \dfrac{s}{r}$

We need to convert the angle $220°$ to radians.

$$220° \cdot \frac{\pi}{180°} = 11 \cdot \frac{\pi}{9} = \frac{11\pi}{9}$$

Now, substitute all known information into the formula.

$$\frac{11\pi}{9} = \frac{s}{15} \;\Rightarrow\; \frac{11\pi}{3} = \frac{s}{5} \;\Rightarrow\; 3s = 55\pi \;\Rightarrow\; s = \frac{55\pi}{3}$$

**Answer:** $\dfrac{55\pi}{3}$ feet

**<u>Example</u>** If a central angle of $155°$ intercepts an arc of length 12 meters, then find the radius of the circle.

We need to convert the angle $155°$ to radians.

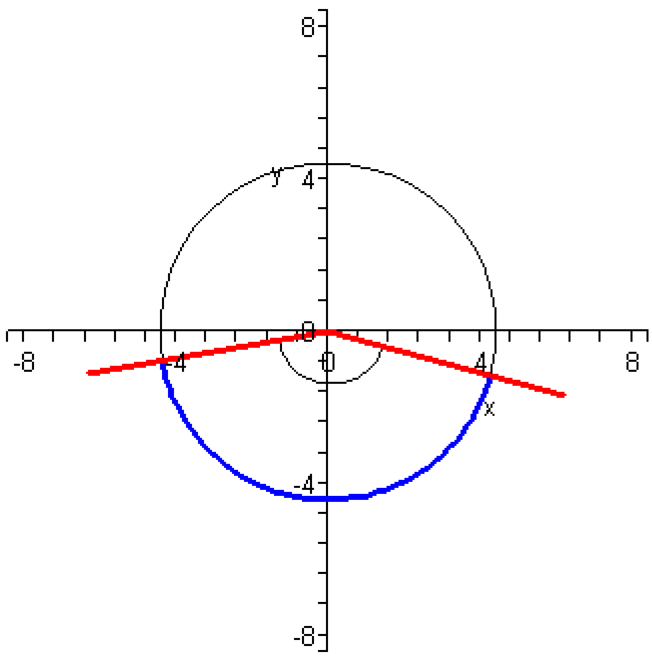
$$155^\circ \cdot \frac{\pi}{180^\circ} = 31 \cdot \frac{\pi}{36} = \frac{31\pi}{36}$$

Now, substitute all known information into the Radian Angle Measurement formula.

$$\frac{31\pi}{36} = \frac{12}{r} \Rightarrow 31\pi r = 432 \Rightarrow r = \frac{432}{31\pi}$$

**Answer:** $\frac{432}{31\pi}$ meters

An illustration of the this central angle, arc length, and radius might be the following:

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