LESSON 11 PYTHAGOREAN AND BASIC IDENTITIES

Examples Use **one** Pythagorean Identity and Basic Identities to find the exact value of the other five trigonometric functions for the following.

1. $\csc \alpha = -5$ and α is in the III quadrant
2. $\cos \beta = \frac{2}{3}$ and $\csc \beta < 0$
3. $\tan \theta = -\frac{\sqrt{17}}{8}$ and $\sin \theta > 0$

Examples Find all the exact solutions for the following equations.

1. $2\cos^2 \theta = 1 + \sin \theta$
2. $7 - 2\cos \alpha = 8\sin^2 \alpha$
3. $\sec^2 \beta = 9 - 7\tan \beta$
4. $-5\cot^2 x = 6\csc x + 5$

Recall the following Basic Identities from Lesson 2:

1. $\tan \theta = \frac{\sin \theta}{\cos \theta}$
2. $\cot \theta = \frac{\cos \theta}{\sin \theta}$
3. $\cot \theta = \frac{1}{\tan \theta}$
4. $\sec \theta = \frac{1}{\cos \theta}$
5. $\csc \theta = \frac{1}{\sin \theta}$

Recall the following Pythagorean Identities from Lesson 2:

1. $\cos^2 \theta + \sin^2 \theta = 1$
2. $\sec^2 \theta - \tan^2 \theta = 1$
3. $\csc^2 \theta - \cot^2 \theta = 1$

Examples Use **one** Pythagorean Identity and Basic Identities to find the exact value of the other five trigonometric functions for the following.

1. $\csc \alpha = -5$ and α is in the III quadrant Back to Examples List

Since $\csc \alpha = -5$, then $\sin \alpha = -\frac{1}{5}$

Since $\csc \alpha = -5$ and $\csc^2 \alpha - \cot^2 \alpha = 1$ by one of the Pythagorean Identities, then

$(-5)^2 - \cot^2 \alpha = 1 \Rightarrow \quad 25 - \cot^2 \alpha = 1 \Rightarrow \quad \cot^2 \alpha = 24 \Rightarrow$

$\cot \alpha = \pm \sqrt{24}$. Since α is in the III quadrant, then $\cot \alpha = \sqrt{24}$.

Since $\cot \alpha = \sqrt{24}$, then $\tan \alpha = \frac{1}{\sqrt{24}}$.

Now, we need to find $\cos \alpha$ and $\sec \alpha$. Since $\sin \alpha = -\frac{1}{5}$ and $\cos^2 \alpha + \sin^2 \alpha = 1$ by one of the Pythagorean Identities, then you **could** use this identity to find $\cos \alpha$. However, you only need to use the Pythagorean Identities once to solve these problems.

Since $\cot \alpha = \frac{\cos \alpha}{\sin \alpha}$, then $\cos \alpha = \cot \alpha \sin \alpha$. Since $\cot \alpha = \sqrt{24}$ and $\sin \alpha = -\frac{1}{5}$, then $\cos \alpha = \sqrt{24}\left(-\frac{1}{5}\right) = -\frac{\sqrt{24}}{5}$.

Since $\cos \alpha = -\frac{\sqrt{24}}{5}$, then $\sec \alpha = -\frac{5}{\sqrt{24}}$.

Answers: $\cos\alpha = -\frac{\sqrt{24}}{5}$, $\sin\alpha = -\frac{1}{5}$, $\tan\alpha = \frac{1}{\sqrt{24}}$, $\sec\alpha = -\frac{5}{\sqrt{24}}$, and $\cot\alpha = \sqrt{24}$

2. $\cos\beta = \frac{2}{3}$ and $\csc\beta < 0$ Back to Examples List

First, determine what quadrant the angle β is in. Using Method 1 from Lesson 6, we have:

$\cos\beta > 0 \Rightarrow$ the x-coordinate of the point of intersection of the terminal side of the angle β with the Unit Circle is positive. That is, $x > 0$.
$\csc\beta < 0 \Rightarrow \sin\beta < 0 \Rightarrow$ the y-coordinate of the point of intersection of the terminal side of the angle β with the Unit Circle is negative. That is, $y < 0$.

Thus, we have that $x > 0$ and $y < 0$. Thus, the angle β is in the IV quadrant. You may use Method 2 or Method 3 from Lesson 6 if you wish.

Since $\cos\beta = \frac{2}{3}$, then $\sec\beta = \frac{3}{2}$

Since $\cos\beta = \frac{2}{3}$ and $\cos^2\beta + \sin^2\beta = 1$ by one of the Pythagorean Identities, then

$$\left(\frac{2}{3}\right)^2 + \sin^2\beta = 1 \Rightarrow \frac{4}{9} + \sin^2\beta = 1 \Rightarrow \sin^2\beta = \frac{5}{9} \Rightarrow$$

$\sin\beta = \pm\frac{\sqrt{5}}{3}$. Since β is in the IV quadrant, then $\sin\beta = -\frac{\sqrt{5}}{3}$.

Since $\sin\beta = -\frac{\sqrt{5}}{3}$, then $\csc\beta = -\frac{3}{\sqrt{5}}$.

Now, we need to find $\tan\beta$ and $\cot\beta$. Since $\tan\beta = \dfrac{\sin\beta}{\cos\beta}$ and $\sin\beta = -\dfrac{\sqrt{5}}{3}$ and $\cos\beta = \dfrac{2}{3}$, then $\tan\beta = \dfrac{-\dfrac{\sqrt{5}}{3}}{\dfrac{2}{3}} = -\dfrac{\sqrt{5}}{2}$.

Since $\tan\beta = -\dfrac{\sqrt{5}}{2}$, then $\cot\beta = -\dfrac{2}{\sqrt{5}}$.

Answers: $\sin\beta = -\dfrac{\sqrt{5}}{3}$, $\tan\beta = -\dfrac{\sqrt{5}}{2}$, $\sec\beta = \dfrac{3}{2}$, $\csc\beta = -\dfrac{3}{\sqrt{5}}$, and $\cot\beta = -\dfrac{2}{\sqrt{5}}$

3. $\tan\theta = -\dfrac{\sqrt{17}}{8}$ and $\sin\theta > 0$

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First, determine what quadrant the angle θ is in. Using Method 1 from Lesson 6, we have:

$\sin\theta > 0 \Rightarrow$ the y-coordinate of the point of intersection of the terminal side of the angle θ with the Unit Circle is positive. That is, $y > 0$.

The tangent of the angle θ is the y-coordinate of the point of intersection of the terminal side of the angle θ with the Unit Circle divided by the x-coordinate of the point of intersection. That is, $\tan\theta = \dfrac{y}{x}$. Since we have that $\tan\theta < 0$ and $y > 0$, then we have the following:

$$(-) = \tan\theta = \frac{y}{x} = \frac{(+)}{?} \Rightarrow x < 0$$

Thus, we have that $x < 0$ and $y > 0$. Thus, the angle θ is in the II quadrant. You may use Method 2 or Method 3 from Lesson 6 if you wish.

Since $\tan\theta = -\dfrac{\sqrt{17}}{8}$, then $\cot\theta = -\dfrac{8}{\sqrt{17}}$

Since $\tan\theta = -\dfrac{\sqrt{17}}{8}$ and $\sec^2\theta - \tan^2\theta = 1$ by one of the Pythagorean Identities, then

$$\sec^2\theta - \left(-\frac{\sqrt{17}}{8}\right)^2 = 1 \Rightarrow \sec^2\theta - \frac{17}{64} = 1 \Rightarrow \sec^2\theta = \frac{81}{64} \Rightarrow$$

$\sec\theta = \pm\dfrac{9}{8}$. Since θ is in the II quadrant, then $\sec\theta = -\dfrac{9}{8}$.

Since $\sec\theta = -\dfrac{9}{8}$, then $\cos\theta = -\dfrac{8}{9}$.

Now, we need to find $\sin\theta$ and $\csc\theta$. Since $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$, then $\sin\theta = \tan\theta\cos\theta$. Since $\tan\theta = -\dfrac{\sqrt{17}}{8}$ and $\cos\theta = -\dfrac{8}{9}$, then $\sin\theta = -\dfrac{\sqrt{17}}{8}\left(-\dfrac{8}{9}\right) = \dfrac{\sqrt{17}}{9}$.

Since $\sin\theta = \dfrac{\sqrt{17}}{9}$, then $\csc\theta = \dfrac{9}{\sqrt{17}}$.

Answers: $\cos\theta = -\dfrac{8}{9}$, $\sin\theta = \dfrac{\sqrt{17}}{9}$, $\sec\theta = -\dfrac{9}{8}$, $\csc\theta = \dfrac{9}{\sqrt{17}}$, and $\cot\theta = -\dfrac{8}{\sqrt{17}}$

<u>Examples</u> Find all the exact solutions for the following equations.

1. $2\cos^2\theta = 1 + \sin\theta$ Back to Examples List

This equation is neither quadratic in $\cos\theta$ nor $\sin\theta$. We will use the Pythagorean Identity $\cos^2\theta + \sin^2\theta = 1$ to replace $\cos^2\theta$ by $1 - \sin^2\theta$. The resulting equation will be quadratic in $\sin\theta$. Since $\cos^2\theta + \sin^2\theta = 1$, then $\cos^2\theta = 1 - \sin^2\theta$. Thus,

$$2\cos^2\theta = 1 + \sin\theta \Rightarrow 2(1 - \sin^2\theta) = 1 + \sin\theta \Rightarrow$$

$$2 - 2\sin^2\theta = 1 + \sin\theta \Rightarrow 0 = 2\sin^2\theta + \sin\theta - 1 \Rightarrow$$

$$(\sin\theta + 1)(2\sin\theta - 1) = 0$$

Thus, either $\sin\theta + 1 = 0$ or $2\sin\theta - 1 = 0$. Thus, there are **two** equations to be solved. NOTE: These types of equations were solved in Lesson 10. Please read that lesson in order to see how to solve these equations.

Since $\sin\theta + 1 = 0 \Rightarrow \sin\theta = -1$, then the solutions of the first equation are $\theta = \dfrac{3\pi}{2} + 2n\pi$, where *n* is an integer.

Since $2\sin\theta - 1 = 0 \Rightarrow \sin\theta = \dfrac{1}{2}$, then the solutions of the second equation are $\theta = \dfrac{\pi}{6} + 2n\pi$ and $\theta = \dfrac{5\pi}{6} + 2n\pi$, where *n* is an integer.

Answers: $\theta = \frac{\pi}{6} + 2n\pi$, $\theta = \frac{5\pi}{6} + 2n\pi$, and $\theta = \frac{3\pi}{2} + 2n\pi$, where n is an integer

2. $7 - 2\cos\alpha = 8\sin^2\alpha$ Back to Examples List

This equation is neither quadratic in $\cos\alpha$ nor $\sin\alpha$. We will use the Pythagorean Identity $\cos^2\alpha + \sin^2\alpha = 1$ to replace $\sin^2\alpha$ by $1 - \cos^2\alpha$. The resulting equation will be quadratic in $\cos\alpha$. Since $\cos^2\alpha + \sin^2\alpha = 1$, then $\sin^2\alpha = 1 - \cos^2\alpha$. Thus,

$$2\cos\alpha + 7 = 8\sin^2\alpha \Rightarrow 2\cos\alpha + 7 = 8(1 - \cos^2\alpha) \Rightarrow$$

$$2\cos\alpha + 7 = 8 - 8\cos^2\alpha \Rightarrow 8\cos^2\alpha + 2\cos\alpha - 1 = 0 \Rightarrow$$

$$(2\cos\alpha + 1)(4\cos\alpha - 1) = 0$$

Thus, either $2\cos\alpha + 1 = 0$ or $4\cos\alpha - 1 = 0$. Thus, there are **two** equations to be solved. NOTE: These types of equations were solved in Lesson 10. Please read that lesson in order to see how to solve these equations.

Since $2\cos\alpha + 1 = 0 \Rightarrow \cos\alpha = -\frac{1}{2}$, then the solutions of the second equation are $\alpha = \frac{2\pi}{3} + 2n\pi$ and $\alpha = \frac{4\pi}{3} + 2n\pi$, where n is an integer.

To solve the second equation $4\cos\alpha - 1 = 0$:

$$4\cos\alpha - 1 = 0 \Rightarrow 4\cos\alpha = 1 \Rightarrow \cos\alpha = \frac{1}{4}$$

Determine where the solutions α will occur. Since $\frac{1}{4}$ is not the maximum positive number for the cosine function, then the solutions do not occur at the coordinate axes. Thus, the solutions occur in two of the four quadrants. Since cosine is positive in the I and IV quadrants, the solutions for this equation occur in those quadrants.

Find the reference angle α' for the solutions α:

$$\cos\alpha = \frac{1}{4} \Rightarrow \cos\alpha' = \frac{1}{4} \Rightarrow \alpha' = \cos^{-1}\frac{1}{4}$$

The solutions in the I quadrant: The one solution in the I quadrant, that is between 0 and 2π, is $\alpha = \cos^{-1}\frac{1}{4}$. Now, all the other solutions in the I quadrant are coterminal to this one solution. Thus, all the solutions in the I quadrant are given by $\alpha = \cos^{-1}\frac{1}{4} + 2n\pi$, where *n* is an integer.

The solutions in the IV quadrant: The one solution in the IV quadrant, that is between -2π and 0, is $\alpha = -\cos^{-1}\frac{1}{4}$. Now, all the other solutions in the IV quadrant are coterminal to this one solution. Thus, all the solutions in the IV quadrant are given by $\alpha = -\cos^{-1}\frac{1}{4} + 2n\pi$, where *n* is an integer.

Thus, the solutions of the second equation $4\cos\alpha - 1 = 0$ are $\alpha = \cos^{-1}\frac{1}{4} + 2n\pi$ and $\alpha = -\cos^{-1}\frac{1}{4} + 2n\pi$, where *n* is an integer.

Answers: $\alpha = \cos^{-1}\frac{1}{4} + 2n\pi$, $\alpha = \frac{2\pi}{3} + 2n\pi$, $\alpha = \frac{4\pi}{3} + 2n\pi$ and $\alpha = -\cos^{-1}\frac{1}{4} + 2n\pi$, where *n* is an integer

3. $\sec^2\beta = 9 - 7\tan\beta$ Back to Examples List

This equation is neither quadratic in $\sec\beta$ nor $\tan\beta$. We will use the Pythagorean Identity $\sec^2\beta - \tan^2\beta = 1$ to replace $\sec^2\beta$ by $1 + \tan^2\beta$. The resulting equation will be quadratic in $\tan\beta$. Since $\sec^2\beta - \tan^2\beta = 1$, then $\sec^2\beta = 1 + \tan^2\beta$. Thus,

$$\sec^2\beta = 9 - 7\tan\beta \Rightarrow 1 + \tan^2\beta = 9 - 7\tan\beta \Rightarrow$$

$$\tan^2\beta + 7\tan\beta - 8 = 0 \Rightarrow (\tan\beta - 1)(\tan\beta + 8) = 0$$

Thus, either $\tan\beta - 1 = 0$ or $\tan\beta + 8 = 0$. Thus, there are **two** equations to be solved. NOTE: These types of equations were solved in Lesson 10. Please read that lesson in order to see how to solve these equations.

Since $\tan\beta - 1 = 0 \Rightarrow \tan\theta = 1$, then the solutions of the first equation are $\beta = \dfrac{\pi}{4} + 2n\pi$ and $\beta = \dfrac{5\pi}{4} + 2n\pi$, where *n* is an integer.

To solve the second equation $\tan\beta + 8 = 0$:

$$\tan\beta + 8 = 0 \Rightarrow \tan\beta = -8$$

Determine where the solutions β will occur. Since tangent is negative in the II and IV quadrants, the solutions for this equation occur in those quadrants.

Find the reference angle β' for the solutions β:

$$\tan\beta = -8 \Rightarrow \tan\beta' = 8 \Rightarrow \beta' = \tan^{-1} 8$$

The solutions in the II quadrant: The one solution in the II quadrant, that is between 0 and 2π, is $\beta = \pi - \beta' = \pi - \tan^{-1} 8$. Now, all the other solutions in the II quadrant are coterminal to this one solution. Thus, all the solutions in the II quadrant are given by $\beta = \pi - \tan^{-1} 8 + 2n\pi = - \tan^{-1} 8 + (2n + 1)\pi$, where n is an integer.

The solutions in the IV quadrant: The one solution in the IV quadrant, that is between -2π and 0, is $\beta = -\tan^{-1} 8$. Now, all the other solutions in the IV quadrant are coterminal to this one solution. Thus, all the solutions in the IV quadrant are given by $\beta = -\tan^{-1} 8 + 2n\pi$, where n is an integer.

Thus, the solutions of the second equation $\tan \beta + 8 = 0$ are $\beta = -\tan^{-1} 8 + (2n + 1)\pi$ and $\beta = -\tan^{-1} 8 + 2n\pi$, where n is an integer. NOTE: Since the period of the tangent function is π, then this answer may also be written as $\beta = -\tan^{-1} 8 + n\pi$, where n is an integer.

Answers: $\beta = \dfrac{\pi}{4} + 2n\pi$, $\beta = \dfrac{5\pi}{4} + 2n\pi$, and $\beta = -\tan^{-1} 8 + n\pi$, where n is an integer

4. $-5\cot^2 x = 6\csc x + 5$ Back to Examples List

This equation is neither quadratic in $\cot x$ nor $\csc x$. We will use the Pythagorean Identity $\csc^2 x - \cot^2 x = 1$ to replace $\cot^2 x$ by $\csc^2 x - 1$. The resulting equation will be quadratic in $\csc x$. Since $\csc^2 x - \cot^2 x = 1$, then $\cot^2 x = \csc^2 x - 1$. Thus,

$$-5\cot^2 x = 6\csc x + 5 \Rightarrow -5(\csc^2 x - 1) = 6\csc x + 5 \Rightarrow$$

$$-5\csc^2 x + 5 = 6\csc x + 5 \Rightarrow -5\csc^2 x = 6\csc x \Rightarrow$$

$$0 = 5\csc^2 x + 6\csc x \Rightarrow \csc x(5\csc x + 6) = 0$$

Thus, either $\csc x = 0$ or $5\csc x + 6 = 0$. Thus, there are **two** equations to be solved.

To solve the first equation $\csc x = 0$: Since $\csc x = 0$ and $\csc x = \frac{1}{\sin x}$, then $\frac{1}{\sin x} = 0$. A fraction is zero if and only if its numerator is zero. The numerator of the fraction $\frac{1}{\sin x}$ will always be 1. Thus, the equation $\csc x = 0$ has no solutions.

To solve the second equation $5\csc x + 6 = 0$:

$$5\csc x + 6 = 0 \Rightarrow 5\csc x = -6 \Rightarrow \csc x = -\frac{6}{5} \Rightarrow \sin x = -\frac{5}{6}$$

Determine where the solutions x will occur. Since $-\frac{5}{6}$ is not the minimum negative number for the sine function, then the solutions do not occur at the coordinate axes. Thus, the solutions occur in two of the four quadrants. Since sine is negative in the III and IV quadrants, the solutions for this equation occur in those quadrants.

Find the reference angle x' for the solutions x:

$$\sin x = -\frac{5}{6} \Rightarrow \sin x' = \frac{5}{6} \Rightarrow x' = \sin^{-1}\frac{5}{6}$$

The solutions in the III quadrant: The one solution in the III quadrant, that is between 0 and 2π, is $x = \pi + x' = \pi + \sin^{-1}\frac{5}{6}$. Now, all the other solutions in the III quadrant are coterminal to this one solution. Thus, all the solutions in the III quadrant are given by $x = \pi + \sin^{-1}\frac{5}{6} + 2n\pi$ = $x = \sin^{-1}\frac{5}{6} + (2n + 1)\pi$, where n is an integer.

The solutions in the IV quadrant: The one solution in the IV quadrant, that is between -2π and 0, is $x = -\sin^{-1}\frac{5}{6}$. Now, all the other solutions in the IV quadrant are coterminal to this one solution. Thus, all the solutions in the IV quadrant are given by $x = -\sin^{-1}\frac{5}{6} + 2n\pi$, where n is an integer.

Thus, the solutions of the second equation $5\csc x + 6 = 0$ are $x = \sin^{-1}\frac{5}{6} + (2n+1)\pi$ and $x = -\sin^{-1}\frac{5}{6} + 2n\pi$, where n is an integer.

Answers: $x = \sin^{-1}\frac{5}{6} + (2n+1)\pi$ and $x = -\sin^{-1}\frac{5}{6} + 2n\pi$, where n is an integer