LESSON 12 SUM AND DIFFERENCE FORMULAS

The sum and difference formulas for the cosine function:

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

The sum and difference formulas for the sine function:

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

The sum and difference formulas for the tangent function:

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

Since $75° = 30° + 45°$, then we can find the exact value of the cosine, sine, and tangent of $75°$ using the respective sum formula with $30°$ and $45°$. Since $30° = \frac{\pi}{6}$ and $45° = \frac{\pi}{4}$, then $75° = \frac{\pi}{6} + \frac{\pi}{4} = \frac{2\pi}{12} + \frac{3\pi}{12} = \frac{5\pi}{12}$. Of course, we could have obtained this by converting $75°$ to units of radians using the conversion factor $\frac{\pi}{180°}$. That is $75° = 75° \cdot \frac{\pi}{180°} = 75 \cdot \frac{\pi}{180} = 15 \cdot \frac{\pi}{36} = 5 \cdot \frac{\pi}{12} = \frac{5\pi}{12}$.

The cosine of $75°$ or $\frac{5\pi}{12}$:

$$\cos 75° = \cos(30° + 45°) = \cos 30° \cos 45° - \sin 30° \sin 45° =$$

$$\frac{\sqrt{3}}{2}\left(\frac{\sqrt{2}}{2}\right) - \frac{1}{2}\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} = \frac{\sqrt{6} - \sqrt{2}}{4} \quad \ldots\ldots \text{ (a)}$$

$$\cos \frac{5\pi}{12} = \cos\left(\frac{\pi}{6} + \frac{\pi}{4}\right) = \cos \frac{\pi}{6} \cos \frac{\pi}{4} - \sin \frac{\pi}{6} \sin \frac{\pi}{4} =$$

$$\frac{\sqrt{3}}{2}\left(\frac{\sqrt{2}}{2}\right) - \frac{1}{2}\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} = \frac{\sqrt{6} - \sqrt{2}}{4} \quad \ldots\ldots \text{ (b)}$$

The sine of $75°$ or $\frac{5\pi}{12}$:

$$\sin 75° = \sin(30° + 45°) = \sin 30° \cos 45° + \cos 30° \sin 45° =$$

$$\frac{1}{2}\left(\frac{\sqrt{2}}{2}\right) + \frac{\sqrt{3}}{2}\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4} = \frac{\sqrt{6} + \sqrt{2}}{4} \quad \ldots\ldots \text{ (c)}$$

$$\sin \frac{5\pi}{12} = \sin\left(\frac{\pi}{6} + \frac{\pi}{4}\right) = \sin \frac{\pi}{6} \cos \frac{\pi}{4} + \cos \frac{\pi}{6} \sin \frac{\pi}{4} =$$

$$\frac{1}{2}\left(\frac{\sqrt{2}}{2}\right) + \frac{\sqrt{3}}{2}\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4} = \frac{\sqrt{6} + \sqrt{2}}{4} \quad \ldots\ldots \text{ (d)}$$

The tangent of $75°$ or $\frac{5\pi}{12}$:

$$\tan 75° = \tan(30° + 45°) = \frac{\tan 30° + \tan 45°}{1 - \tan 30° \tan 45°} = \frac{\frac{\sqrt{3}}{3} + 1}{1 - \frac{\sqrt{3}}{3}(1)} =$$

$$\frac{\frac{\sqrt{3}}{3} + 1}{1 - \frac{\sqrt{3}}{3}} = \frac{\frac{\sqrt{3}}{3} + 1}{1 - \frac{\sqrt{3}}{3}} \cdot \frac{3}{3} = \frac{\sqrt{3} + 3}{3 - \sqrt{3}} = \frac{3 + \sqrt{3}}{3 - \sqrt{3}} =$$

$$\frac{3 + \sqrt{3}}{3 - \sqrt{3}} \cdot \frac{3 + \sqrt{3}}{3 + \sqrt{3}} = \frac{(3 + \sqrt{3})^2}{9 - 3} = \frac{9 + 6\sqrt{3} + 3}{6} = \frac{12 + 6\sqrt{3}}{6} =$$

$$\frac{6(2 + \sqrt{3})}{6} = 2 + \sqrt{3} \quad \ldots\ldots \ (e)$$

$$\tan \frac{5\pi}{12} = \tan\left(\frac{\pi}{6} + \frac{\pi}{4}\right) = \frac{\tan \frac{\pi}{6} + \tan \frac{\pi}{4}}{1 - \tan \frac{\pi}{6} \tan \frac{\pi}{4}} = \frac{\frac{\sqrt{3}}{3} + 1}{1 - \frac{\sqrt{3}}{3}(1)} =$$

$$\frac{\frac{\sqrt{3}}{3} + 1}{1 - \frac{\sqrt{3}}{3}} = \cdots\cdots\cdot = 2 + \sqrt{3} \quad \ldots\ldots \ (f)$$

Since $75°$ is the reference angle for $\pm 75°$, $\pm 105°$, $\pm 255°$, and $\pm 285°$, then we will be able to find the exact value of the six trigonometric functions for these angles.

Since $\frac{5\pi}{12}$ is the reference angle for $\pm\frac{5\pi}{12}$, $\pm\frac{7\pi}{12}$, $\pm\frac{17\pi}{12}$, and $\pm\frac{19\pi}{12}$, then we will be able to find the exact value of the six trigonometric functions for these angles.

Examples Use a reference angle to find the exact value of the six trigonometric functions of the following angles.

1. $\theta = \frac{7\pi}{12}$ (This is the $105°$ angle in units of degrees.)

The angle $\theta = \frac{7\pi}{12}$ is in the II quadrant. The reference angle of the angle $\theta = \frac{7\pi}{12}$ is the angle $\theta' = \frac{5\pi}{12}$.

Since cosine is negative in the II quadrant and $\cos\frac{5\pi}{12} = \frac{\sqrt{6} - \sqrt{2}}{4}$ by (b) above, then

$$\cos\frac{7\pi}{12} = -\cos\frac{5\pi}{12} = -\frac{\sqrt{6} - \sqrt{2}}{4} = \frac{\sqrt{2} - \sqrt{6}}{4}$$

$$\sec\frac{7\pi}{12} = \frac{4}{\sqrt{2} - \sqrt{6}} = \frac{4}{\sqrt{2} - \sqrt{6}} \cdot \frac{\sqrt{2} + \sqrt{6}}{\sqrt{2} + \sqrt{6}} =$$

$$\frac{4(\sqrt{2} + \sqrt{6})}{2 - 6} = \frac{4(\sqrt{2} + \sqrt{6})}{-4} = -(\sqrt{2} + \sqrt{6})$$

Since sine is positive in the II quadrant and $\sin\frac{5\pi}{12} = \frac{\sqrt{6} + \sqrt{2}}{4}$ by (d) above, then

$$\sin\frac{7\pi}{12} = \sin\frac{5\pi}{12} = \frac{\sqrt{6} + \sqrt{2}}{4}$$

$$\csc\frac{7\pi}{12} = \frac{4}{\sqrt{6} + \sqrt{2}} = \frac{4}{\sqrt{6} + \sqrt{2}} \cdot \frac{\sqrt{6} - \sqrt{2}}{\sqrt{6} - \sqrt{2}} =$$

$$\frac{4(\sqrt{6} - \sqrt{2})}{6 - 2} = \frac{4(\sqrt{6} - \sqrt{2})}{4} = \sqrt{6} - \sqrt{2}$$

Since tangent is negative in the II quadrant and $\tan\frac{5\pi}{12} = 2 + \sqrt{3}$ by (f) above, then

$$\tan\frac{7\pi}{12} = -\tan\frac{5\pi}{12} = -(2 + \sqrt{3})$$

$$\cot\frac{7\pi}{12} = -\frac{1}{2 + \sqrt{3}} = -\frac{1}{2 + \sqrt{3}} \cdot \frac{2 - \sqrt{3}}{2 - \sqrt{3}} = -\frac{2 - \sqrt{3}}{4 - 3} =$$

$$-\frac{2 - \sqrt{3}}{1} = -(2 - \sqrt{3}) = \sqrt{3} - 2$$

2. $\alpha = 285°$ (This is the $\frac{19\pi}{12}$ angle in units of radians.)

The angle $\alpha = 285°$ is in the IV quadrant. The reference angle of the angle $\alpha = 285°$ is the angle $\alpha' = 75°$.

Since cosine is positive in the IV quadrant and $\cos 75° = \frac{\sqrt{6} - \sqrt{2}}{4}$ by (a) above, then

$$\cos 285^\circ = \cos 75^\circ = \frac{\sqrt{6} - \sqrt{2}}{4}$$

$$\sec 285^\circ = \frac{4}{\sqrt{6} - \sqrt{2}} = \frac{4}{\sqrt{6} - \sqrt{2}} \cdot \frac{\sqrt{6} + \sqrt{2}}{\sqrt{6} + \sqrt{2}} =$$

$$\frac{4(\sqrt{6} + \sqrt{2})}{6 - 2} = \frac{4(\sqrt{6} + \sqrt{2})}{4} = \sqrt{6} + \sqrt{2}$$

Since sine is negative in the IV quadrant and $\sin 75^\circ = \dfrac{\sqrt{6} + \sqrt{2}}{4}$ by (c) above, then

$$\sin 285^\circ = -\sin 75^\circ = -\frac{\sqrt{6} + \sqrt{2}}{4}$$

$$\csc 285^\circ = -\frac{4}{\sqrt{6} + \sqrt{2}} = -\frac{4}{\sqrt{6} + \sqrt{2}} \cdot \frac{\sqrt{6} - \sqrt{2}}{\sqrt{6} - \sqrt{2}} =$$

$$-\frac{4(\sqrt{6} - \sqrt{2})}{6 - 2} = -\frac{4(\sqrt{6} - \sqrt{2})}{4} = -(\sqrt{6} - \sqrt{2}) =$$

$$\sqrt{2} - \sqrt{6}$$

Since tangent is negative in the IV quadrant and $\tan 75^\circ = 2 + \sqrt{3}$ by (e) above, then

$$\tan 285^\circ = -\tan 75^\circ = -(2 + \sqrt{3})$$

$$\cot 285^\circ = -\frac{1}{2+\sqrt{3}} = -\frac{1}{2+\sqrt{3}} \cdot \frac{2-\sqrt{3}}{2-\sqrt{3}} = -\frac{2-\sqrt{3}}{4-3} =$$

$$-\frac{2-\sqrt{3}}{1} = -(2-\sqrt{3}) = \sqrt{3}-2$$

3. $\beta = -105^\circ$ (This is the $-\frac{7\pi}{12}$ angle in units of radians.)

The angle $\beta = -105^\circ$ is in the III quadrant. The reference angle of the angle $\beta = -105^\circ$ is the angle $\beta' = 75^\circ$.

Since cosine is negative in the III quadrant and $\cos 75^\circ = \frac{\sqrt{6}-\sqrt{2}}{4}$ by (a) above, then

$$\cos(-105^\circ) = -\cos 75^\circ = -\frac{\sqrt{6}-\sqrt{2}}{4} = \frac{\sqrt{2}-\sqrt{6}}{4}$$

$$\sec(-105^\circ) = -\frac{4}{\sqrt{6}-\sqrt{2}} = -\frac{4}{\sqrt{6}-\sqrt{2}} \cdot \frac{\sqrt{6}+\sqrt{2}}{\sqrt{6}+\sqrt{2}} =$$

$$-\frac{4(\sqrt{6}+\sqrt{2})}{6-2} = -\frac{4(\sqrt{6}+\sqrt{2})}{4} = -(\sqrt{6}+\sqrt{2})$$

Since sine is negative in the III quadrant and $\sin 75^\circ = \frac{\sqrt{6}+\sqrt{2}}{4}$ by (c) above, then

$$\sin(-105^\circ) = -\sin 75^\circ = -\frac{\sqrt{6}+\sqrt{2}}{4}$$

$$\csc(-105°) = -\frac{4}{\sqrt{6}+\sqrt{2}} = -\frac{4}{\sqrt{6}+\sqrt{2}} \cdot \frac{\sqrt{6}-\sqrt{2}}{\sqrt{6}-\sqrt{2}} =$$

$$-\frac{4(\sqrt{6}-\sqrt{2})}{6-2} = -\frac{4(\sqrt{6}-\sqrt{2})}{4} = -(\sqrt{6}-\sqrt{2}) =$$

$$\sqrt{2}-\sqrt{6}$$

Since tangent is positive in the III quadrant and $\tan 75° = 2+\sqrt{3}$ by (e) above, then

$$\tan(-105°) = \tan 75° = 2+\sqrt{3}$$

$$\cot(-105°) = \frac{1}{2+\sqrt{3}} = \frac{1}{2+\sqrt{3}} \cdot \frac{2-\sqrt{3}}{2-\sqrt{3}} = \frac{2-\sqrt{3}}{4-3} =$$

$$\frac{2-\sqrt{3}}{1} = 2-\sqrt{3}$$

Since $15° = 60° - 45°$, then we can find the cosine, sine, and tangent of $15°$ using the respective difference formula with $60°$ and $45°$. Since $60° = \frac{\pi}{3}$ and $45° = \frac{\pi}{4}$, then $15° = \frac{\pi}{3} - \frac{\pi}{4} = \frac{4\pi}{12} - \frac{3\pi}{12} = \frac{\pi}{12}$. Of course, we could have obtained this by converting $15°$ to units of radians using the conversion factor $\frac{\pi}{180°}$. That is $15° = 15° \cdot \frac{\pi}{180°} = 15 \cdot \frac{\pi}{180} = 5 \cdot \frac{\pi}{60} = 1 \cdot \frac{\pi}{12} = \frac{\pi}{12}$.

The cosine of $15°$ or $\frac{\pi}{12}$:

$$\cos 15° = \cos(60° - 45°) = \cos 60° \cos 45° + \sin 60° \sin 45° =$$

$$\frac{1}{2}\left(\frac{\sqrt{2}}{2}\right) + \frac{\sqrt{3}}{2}\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4} = \frac{\sqrt{6} + \sqrt{2}}{4} \quad \ldots\ldots \text{ (g)}$$

$$\cos \frac{\pi}{12} = \cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \cos \frac{\pi}{3} \cos \frac{\pi}{4} + \sin \frac{\pi}{3} \sin \frac{\pi}{4} =$$

$$\frac{1}{2}\left(\frac{\sqrt{2}}{2}\right) + \frac{\sqrt{3}}{2}\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4} = \frac{\sqrt{6} + \sqrt{2}}{4} \quad \ldots\ldots \text{ (h)}$$

The sine of $15°$ or $\frac{\pi}{12}$:

$$\sin 15° = \sin(60° - 45°) = \sin 60° \cos 45° - \cos 60° \sin 45° =$$

$$\frac{\sqrt{3}}{2}\left(\frac{\sqrt{2}}{2}\right) - \frac{1}{2}\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} = \frac{\sqrt{6} - \sqrt{2}}{4} \quad \ldots\ldots \text{ (i)}$$

$$\sin \frac{\pi}{12} = \sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \sin \frac{\pi}{3} \cos \frac{\pi}{4} - \cos \frac{\pi}{3} \sin \frac{\pi}{4} =$$

$$\frac{\sqrt{3}}{2}\left(\frac{\sqrt{2}}{2}\right) - \frac{1}{2}\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} = \frac{\sqrt{6} - \sqrt{2}}{4} \quad \ldots\ldots \text{ (j)}$$

The tangent of $15°$ or $\frac{\pi}{12}$:

$$\tan 15° = \tan(60° - 45°) = \frac{\tan 60° - \tan 45°}{1 + \tan 60° \tan 45°} = \frac{\sqrt{3} - 1}{1 + \sqrt{3}\,(1)} =$$

$$\frac{\sqrt{3} - 1}{1 + \sqrt{3}} = \frac{\sqrt{3} - 1}{\sqrt{3} + 1} = \frac{\sqrt{3} - 1}{\sqrt{3} + 1} \cdot \frac{\sqrt{3} - 1}{\sqrt{3} - 1} = \frac{(\sqrt{3} - 1)^2}{3 - 1} =$$

$$\frac{3 - 2\sqrt{3} + 1}{2} = \frac{4 - 2\sqrt{3}}{2} = \frac{2(2 - \sqrt{3})}{2} = 2 - \sqrt{3} \quad \ldots\ldots \text{ (k)}$$

$$\tan\frac{\pi}{12} = \tan\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \frac{\tan\frac{\pi}{3} - \tan\frac{\pi}{4}}{1 + \tan\frac{\pi}{3}\tan\frac{\pi}{4}} = \frac{\sqrt{3} - 1}{1 + \sqrt{3}\,(1)} =$$

$$\frac{\sqrt{3} - 1}{1 + \sqrt{3}} = \frac{\sqrt{3} - 1}{\sqrt{3} + 1} = \cdots\cdots\cdot = 2 - \sqrt{3} \quad \ldots\ldots \text{ (l)}$$

NOTE: We also have that $15° = 45° - 30°$. So, you can find the exact value of the cosine, sine, and tangent of $15°$ using the respective difference formula with $45°$ and $30°$. Of course, you will obtain the same values that were obtained above.

Since $15°$ is the reference angle for $\pm 15°$, $\pm 165°$, $\pm 195°$, and $\pm 345°$, then we will be able to find the exact value of the six trigonometric functions for these angles.

Since $\frac{\pi}{12}$ is the reference angle for $\pm\frac{\pi}{12}$, $\pm\frac{11\pi}{12}$, $\pm\frac{13\pi}{12}$, and $\pm\frac{23\pi}{12}$, then we will be able to find the exact value of the six trigonometric functions for these angles.