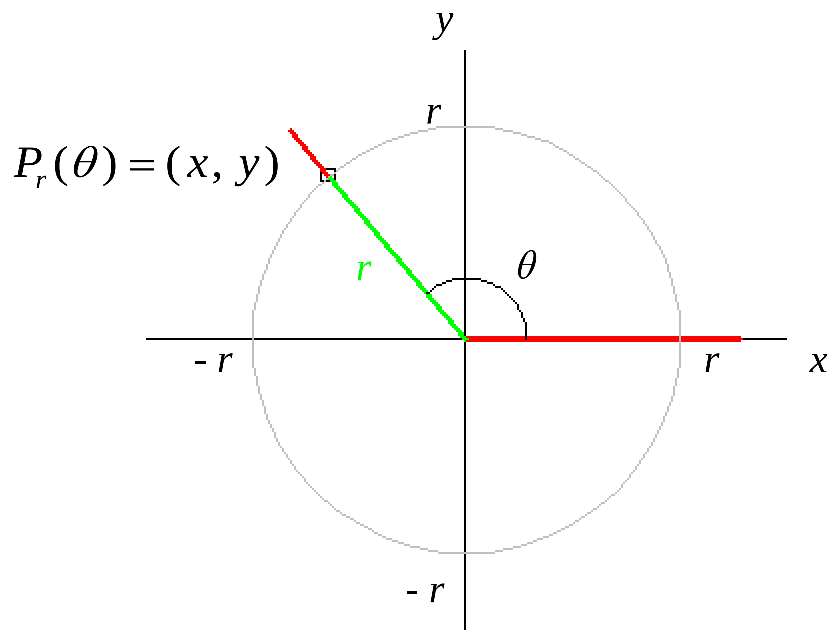
LESSON 5 DEFINITION OF THE SIX TRIGONOMETRIC FUNCTIONS DETERMINED BY A POINT AND A LINE IN THE *xy*-PLANE

Topics in this lesson:

1. DEFINITION AND EXAMPLES OF THE SIX TRIGONOMETRIC FUNCTIONS DETERMINED BY A POINT IN THE *xy*-PLANE



Definition Let $P_r(\theta) = (x, y)$ be a point on the terminal side of the angle θ which is not the origin $(0, 0)$. Then we define the following six trigonometric functions of the angle θ

$\cos\theta = \dfrac{x}{r}$ $\qquad$ $\sec\theta = \dfrac{r}{x}$, provided that $x \neq 0$

$\sin\theta = \dfrac{y}{r}$ $\qquad$ $\csc\theta = \dfrac{r}{y}$, provided that $y \neq 0$

$\tan\theta = \dfrac{y}{x}$, provided that $x \neq 0$ $\qquad$ $\cot\theta = \dfrac{x}{y}$, provided that $y \neq 0$

where $r = \sqrt{x^2 + y^2}$

NOTE: This is exactly the same definition that was given in Section 1 of Lesson 2 . $r = \sqrt{x^2 + y^2}$ is the distance from the point $P_r(\theta) = (x, y)$ to the origin $(0, 0)$, which is the radius of the circle whose equation is $x^2 + y^2 = r^2$. The definition above is saying that in order to define any one of the six trigonometric functions, you only need to know the coordinates of any point on the terminal side of the angle θ, which is not the origin $(0, 0)$, and you do not need the circle. However, hopefully, you would agree that the Unit Circle has been helpful for us and that you would continue to make use of it. I know that I use it all the time to help me.

Notice that for the examples below, we will not have to make use of a reference angle for the problem. The sign of the answer will follow from the definition of the trigonometric function.

<u>Examples</u> Find the exact value of the six trigonometric functions of the angle θ if the given point is on the terminal side of θ.

1. $(-8, 6)$

This point is in the second quadrant and lies on a circle of radius $r = \sqrt{64 + 36} = \sqrt{100} = 10$. Animation of the point and the angle.

$\cos\theta = \dfrac{x}{r} = \dfrac{-8}{10} = -\dfrac{4}{5}$ $\qquad$ $\sec\theta = -\dfrac{5}{4}$

$\sin\theta = \dfrac{y}{r} = \dfrac{6}{10} = \dfrac{3}{5}$ $\qquad$ $\csc\theta = \dfrac{5}{3}$

$\tan\theta = \dfrac{y}{x} = \dfrac{6}{-8} = -\dfrac{3}{4}$ $\qquad$ $\cot\theta = -\dfrac{4}{3}$

NOTE: In Lesson 9, we will find that the angle θ is approximately $143.1°$ or any angle coterminal to this angle.

2. $(\sqrt{3}, -9)$

This point is in the fourth quadrant and lies on a circle of radius $r = \sqrt{3+81} = \sqrt{84} = 2\sqrt{21}$. Animation of the point and the angle.

$$\cos\theta = \frac{x}{r} = \frac{\sqrt{3}}{\sqrt{84}} = \sqrt{\frac{3}{84}} = \sqrt{\frac{1}{28}} = \frac{1}{\sqrt{28}} = \frac{1}{2\sqrt{7}} = \frac{\sqrt{7}}{2(7)} = \frac{\sqrt{7}}{14}$$

$$\cos\theta = \frac{1}{2\sqrt{7}} \Rightarrow \sec\theta = \sqrt{28} = 2\sqrt{7}$$

$$\sin\theta = \frac{y}{r} = -\frac{9}{\sqrt{84}} = -\frac{9}{2\sqrt{21}} = -\frac{9\sqrt{21}}{2(21)} = -\frac{3\sqrt{21}}{2(7)} = -\frac{3\sqrt{21}}{14}$$

$$\sin\theta = -\frac{9}{2\sqrt{21}} \Rightarrow \csc\theta = -\frac{2\sqrt{21}}{9}$$

$$\tan\theta = \frac{y}{x} = \frac{-9}{\sqrt{3}} = -\frac{9\sqrt{3}}{3} = -3\sqrt{3} \qquad \cot\theta = -\frac{1}{3\sqrt{3}} = -\frac{\sqrt{3}}{9}$$

NOTE: In Lesson 9, we will find that the angle θ is approximately $280.9°$ or any angle coterminal to this angle.

3. $(-24, 0)$

This point is on the negative x-axis and lies on a circle of radius $r = \sqrt{(-24)^2 + 0} = \sqrt{24^2} = 24$. Animation of the point and the angle.

$$\cos\theta = \frac{x}{r} = \frac{-24}{24} = -1 \qquad \sec\theta = -1$$

$$\sin\theta = \frac{y}{r} = \frac{0}{24} = 0 \qquad \csc\theta = \text{undefined}$$

$$\tan\theta = \frac{y}{x} = \frac{0}{-24} = 0 \qquad \cot\theta = \text{undefined}$$

NOTE: We already know that the angle θ is 180° or any angle coterminal to this angle.

4. $(-12, -18)$

This point is in the third quadrant and lies on a circle of radius $r = \sqrt{(-12)^2 + (-18)^2} = \sqrt{12^2 + 18^2} = \sqrt{(6 \cdot 2)^2 + (6 \cdot 3)^2} = \sqrt{6^2 4 + 6^2 9} = \sqrt{6^2(4+9)} = 6\sqrt{13}$. Animation of the point and the angle.

$$\cos\theta = \frac{x}{r} = \frac{-12}{6\sqrt{13}} = -\frac{2}{\sqrt{13}} \qquad \sec\theta = -\frac{\sqrt{13}}{2}$$

$$\sin\theta = \frac{y}{r} = \frac{-18}{6\sqrt{13}} = -\frac{3}{\sqrt{13}} \qquad \csc\theta = -\frac{\sqrt{13}}{3}$$

$$\tan\theta = \frac{y}{x} = \frac{-18}{-12} = \frac{3}{2} \qquad \cot\theta = \frac{2}{3}$$

NOTE: In Lesson 9, we will find that the angle θ is approximately 236.3° or any angle coterminal to this angle.

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2. EXAMPLES OF THE SIX TRIGONOMETRIC FUNCTIONS DETERMINED BY A LINE IN THE xy-PLANE

Example The terminal side of the angle α is in the fourth quadrant and lies on the line $y = -\dfrac{3}{7}x$. Find the exact value of the six trigonometric functions of the angle α.

Pick a point on the given line that lies in the fourth quadrant. Since *x*-coordinates are positive in the fourth quadrant, you will need to choose a positive number for the *x*-coordinate of the point. If $x = 7$, then $y = -\dfrac{3}{7}(7) = -3$. Thus, the point $(7, -3)$ lies on the line $y = -\dfrac{3}{7}x$ in the fourth quadrant. Since the terminal side of α lies on the line $y = -\dfrac{3}{7}x$, then the point $(7, -3)$ lies on the terminal side of α. This point lies on a circle of radius $r = \sqrt{49 + 9} = \sqrt{58}$. Animation of the line, point, and the angle.

$$\cos\alpha = \frac{7}{\sqrt{58}} \qquad \sec\alpha = \frac{\sqrt{58}}{7}$$

$$\sin\alpha = -\frac{3}{\sqrt{58}} \qquad \csc\alpha = -\frac{\sqrt{58}}{3}$$

$$\tan\alpha = -\frac{3}{7} \qquad \cot\alpha = -\frac{7}{3}$$

NOTE: In Lesson 9, we will find that the angle α is approximately $336.8°$ or any angle coterminal to this angle.

Question: What was the slope of the given line $y = -\dfrac{3}{7}x$? Answer: $-\dfrac{3}{7}$

What was the tangent of the angle α? Answer: $-\frac{3}{7}$

Example The terminal side of the angle β is in the third quadrant and lies on the line $10x - 8y = 0$. Find the exact value of the six trigonometric functions of the angle β.

First, take the equation for the line and solve for *y*.

$$10x - 8y = 0 \;\Rightarrow\; 10x = 8y \;\Rightarrow\; y = \frac{10}{8}x \;\Rightarrow\; y = \frac{5}{4}x$$

Pick a point on this line that lies in the third quadrant. Since *x*-coordinates are negative in the third quadrant, you will need to choose a negative number for the *x*-coordinate of the point. If $x = -4$, then $y = \frac{5}{4}(-4) = -5$. Thus, the point $(-4, -5)$ lies on the line $y = \frac{5}{4}x$ in the third quadrant. Since the terminal side of β lies on the line $y = \frac{5}{4}x$, then the point $(-4, -5)$ lies on the terminal side of β. This point lies on a circle of radius $r = \sqrt{16 + 25} = \sqrt{41}$. Animation of the line, point, and the angle.

$\cos\beta = -\frac{4}{\sqrt{41}}$ $\qquad$ $\sec\beta = -\frac{\sqrt{41}}{4}$

$\sin\beta = -\frac{5}{\sqrt{41}}$ $\qquad$ $\csc\beta = -\frac{\sqrt{41}}{5}$

$\tan\beta = \frac{5}{4}$ $\qquad$ $\cot\beta = \frac{4}{5}$

NOTE: In Lesson 9, we will find that the angle β is approximately 231.3° or any angle coterminal to this angle.

Question: What was the slope of the line $y = \frac{5}{4}x$? Answer: $\frac{5}{4}$

What was the tangent of the angle β? Answer: $\frac{5}{4}$

Example The terminal side of the angle γ is in the second quadrant and lies on the line $y = -4x$. Find the exact value of the six trigonometric functions of the angle γ.

Pick a point on the given line that lies in the second quadrant. Since *x*-coordinates are negative in the second quadrant, you will need to choose a negative number for the *x*-coordinate of the point. If $x = -1$, then $y = -4(-1) = 4$. Thus, the point $(-1, 4)$ lies on the line $y = -4x$ in the second quadrant. Since the terminal side of γ lies on the line $y = -4x$, then the point $(-1, 4)$ lies on the terminal side of γ. This point lies on a circle of radius $r = \sqrt{1 + 16} = \sqrt{17}$. Animation of the line, point, and the angle.

$$\cos \gamma = -\frac{1}{\sqrt{17}} \qquad \sec \gamma = -\sqrt{17}$$

$$\sin \gamma = \frac{4}{\sqrt{17}} \qquad \csc \gamma = \frac{\sqrt{17}}{4}$$

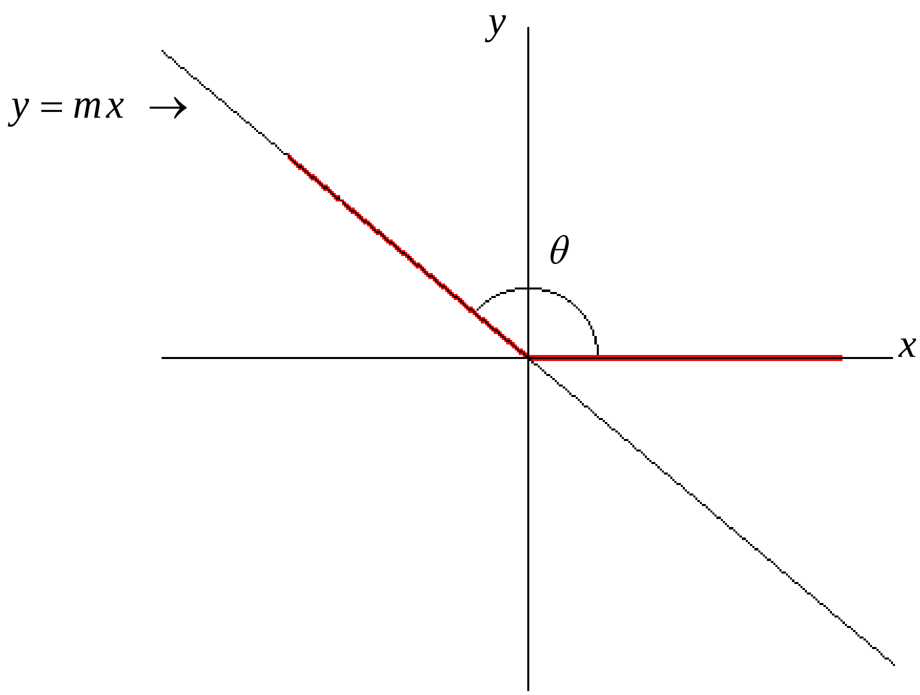
$$\tan \gamma = -4 \qquad \cot \gamma = -\frac{1}{4}$$

NOTE: In Lesson 9, we will find that the angle γ is approximately 104.0° or any angle coterminal to this angle.

Question: What was the slope of the given line $y = -4x$? Answer: -4

What was the tangent of the angle γ? Answer: -4

Given the graph of the line $y = mx$ and the angle θ below, then $\tan\theta = m$. This will be proved in the next lesson.



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