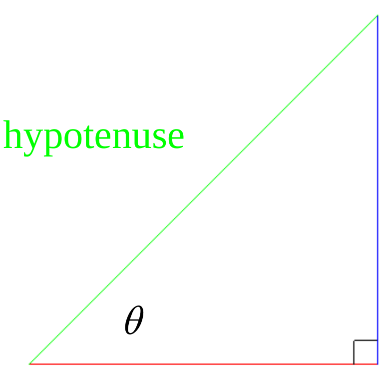
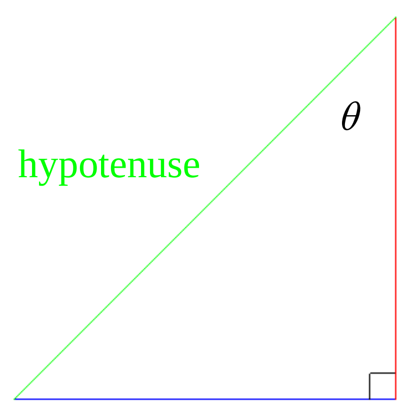
LESSON 6 THE SIX TRIGONOMETRIC FUNCTIONS IN TERMS OF A RIGHT TRIANGLE

Topics in this lesson:

1. DEFINITION AND EXAMPLES OF THE TRIGONOMETRIC FUNCTIONS OF AN ACUTE ANGLE IN TERMS OF A RIGHT TRIANGLE
2. USING A RIGHT TRIANGLE TO FIND THE VALUE OF THE SIX TRIGONOMETRIC FUNCTIONS OF ANGLES IN THE FIRST, SECOND, THIRD, AND FOURTH QUADRANTS

1. DEFINITION AND EXAMPLES OF THE TRIGONOMETRIC FUNCTIONS OF AN ACUTE ANGLE IN TERMS OF A RIGHT TRIANGLE





<u>Definition</u> Given the angle θ in the triangle above. We define the following

$$\cos\theta = \frac{adj}{hyp} \qquad \sec\theta = \frac{hyp}{adj}$$

$$\sin\theta = \frac{opp}{hyp} \qquad \csc\theta = \frac{hyp}{opp}$$

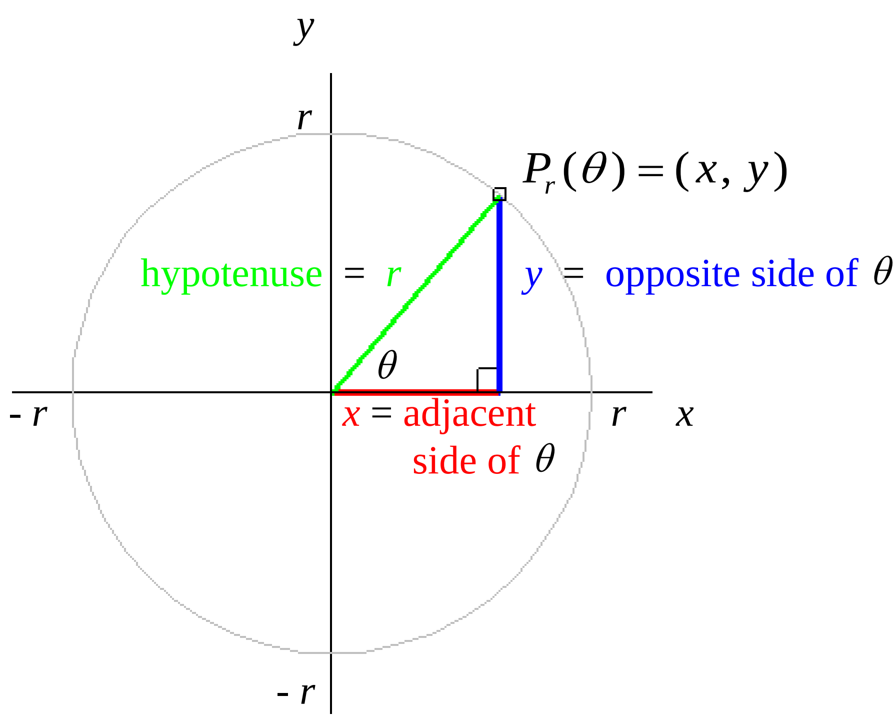
$$\tan\theta = \frac{opp}{adj} \qquad \cot\theta = \frac{adj}{opp}$$

Illustration of the definition of the cosine function, sine function, tangent function, secant function, cosecant function, and cotangent function for the acute angle θ using right triangle trigonometry.

Second illustration of the cosine function, sine function, tangent function, secant function, cosecant function, and cotangent function for the acute angle θ using right triangle trigonometry.

Illustration of the definition of all the six trigonometric functions for an acute angle θ using right triangle trigonometry. Second illustration of all the six trigonometric functions.

NOTE: Since the three angles of any triangle sum to $180°$ and the right angle in the triangle is $90°$, then the other two angles in the right triangle must sum to $90°$. Thus, the other two angles in the triangle must be greater than $0°$ and less than $90°$. Thus, the other two angles in the triangle are acute angles. Thus, the angle θ above is an acute angle. If we consider the angle θ in standard position, then θ is in the first quadrant and we would have the following:



$$\cos\theta = \frac{x}{r} = \frac{adj}{hyp} \qquad \sec\theta = \frac{r}{x} = \frac{hyp}{adj}$$

$$\sin\,\theta = \frac{y}{r} = \frac{opp}{hyp} \qquad\qquad \csc\,\theta = \frac{r}{y} = \frac{hyp}{opp}$$

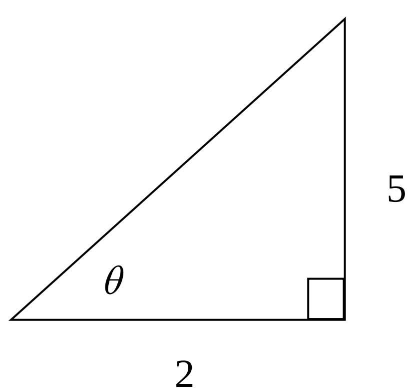
$$\tan\,\theta = \frac{y}{x} = \frac{opp}{adj} \qquad\qquad \cot\,\theta = \frac{x}{y} = \frac{adj}{opp}$$

The advantage to this definition is that the angle θ does not have to be in standard position in order to recognize the hypotenuse of the triangle and the opposite and adjacent side of the angle θ. Thus, the right triangle can be oriented anyway in the plane. The triangle could be spun in the plane and when it stopped spinning, you would still be able to identify the hypotenuse of the triangle and the opposite and adjacent side of the angle θ.

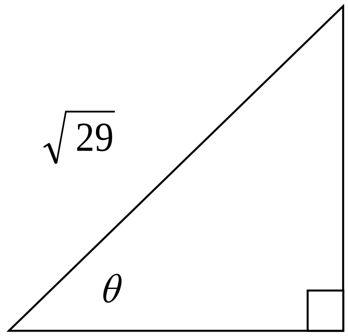
One disadvantage of this definition is that the angle θ must be an acute angle. This would exclude any angle whose terminal side lies on one of the coordinate axes. It would also exclude any angle whose terminal side lies in the second, third, fourth and first (by rotating clockwise) quadrants; however, the reference angle for these angles would be acute and could be put into a right triangle.

Examples Find the exact value of the six trigonometric functions for the following angles.

1.



Using the Pythagorean Theorem to find the length of the hypotenuse, we have that the length of the hypotenuse is $\sqrt{4+25} = \sqrt{29}$. Thus, we have that



5 = opposite side of θ

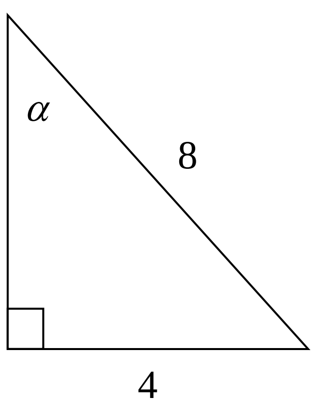
2 = adjacent side of θ

$$\cos\theta = \frac{adj}{hyp} = \frac{2}{\sqrt{29}} \qquad \sec\theta = \frac{hyp}{adj} = \frac{\sqrt{29}}{2}$$

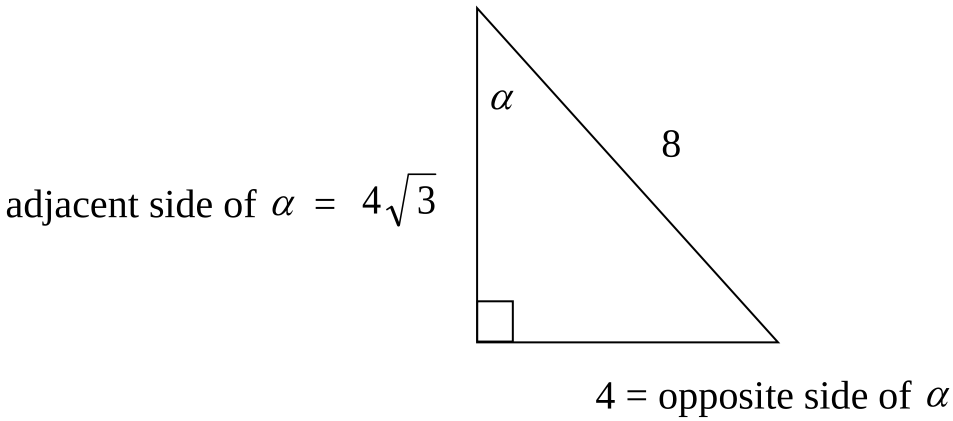
$$\sin\theta = \frac{opp}{hyp} = \frac{5}{\sqrt{29}} \qquad \csc\theta = \frac{hyp}{opp} = \frac{\sqrt{29}}{5}$$

$$\tan\theta = \frac{opp}{adj} = \frac{5}{2} \qquad \cot\theta = \frac{adj}{opp} = \frac{2}{5}$$

2.



Using the Pythagorean Theorem to find the length of the second side, we have that the length of the second side is $\sqrt{64-16} = \sqrt{48} = 4\sqrt{3}$. Thus, we have that



$$\cos\alpha = \frac{adj}{hyp} = \frac{4\sqrt{3}}{8} = \frac{\sqrt{3}}{2} \qquad \sec\alpha = \frac{2}{\sqrt{3}}$$

$$\sin\alpha = \frac{opp}{hyp} = \frac{4}{8} = \frac{1}{2} \qquad \csc\alpha = 2$$

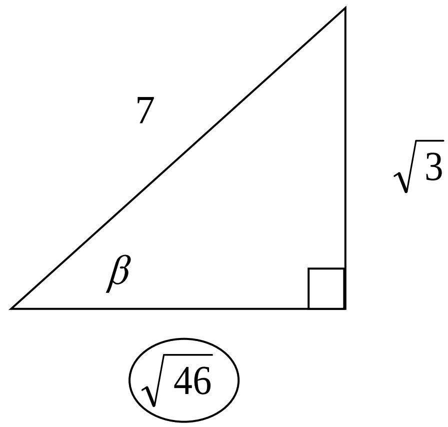
$$\tan\alpha = \frac{opp}{adj} = \frac{4}{4\sqrt{3}} = \frac{1}{\sqrt{3}} \qquad \cot\alpha = \sqrt{3}$$

NOTE: These answers should look familiar to you. The angle α would have to be the 30° or $\frac{\pi}{6}$ angle.

Examples Use a right triangle to find the exact value of the other five trigonometric functions if given the following.

1. $\sin\beta = \frac{\sqrt{3}}{7}$ and β is an acute angle

$$\sin\beta = \frac{\sqrt{3}}{7} = \frac{opp}{hyp}$$



NOTE: The number circled above was found using the Pythagorean Theorem by $\sqrt{49-3} = \sqrt{46}$.

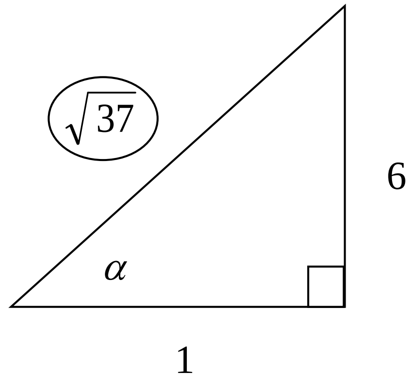
$$\cos\beta = \frac{adj}{hyp} = \frac{\sqrt{46}}{7} \qquad \sec\beta = \frac{7}{\sqrt{46}}$$

$$\sin\beta = \frac{\sqrt{3}}{7} \text{ (given)} \qquad \csc\beta = \frac{7}{\sqrt{3}}$$

$$\tan\beta = \frac{opp}{adj} = \frac{\sqrt{3}}{\sqrt{46}} = \sqrt{\frac{3}{46}} \qquad \cot\beta = \frac{\sqrt{46}}{\sqrt{3}} = \sqrt{\frac{46}{3}}$$

2. $\tan\alpha = 6$ and α is an acute angle

$$\tan\alpha = 6 = \frac{6}{1} = \frac{opp}{adj}$$



NOTE: The number circled above was found using the Pythagorean Theorem by $\sqrt{1+36} = \sqrt{37}$.

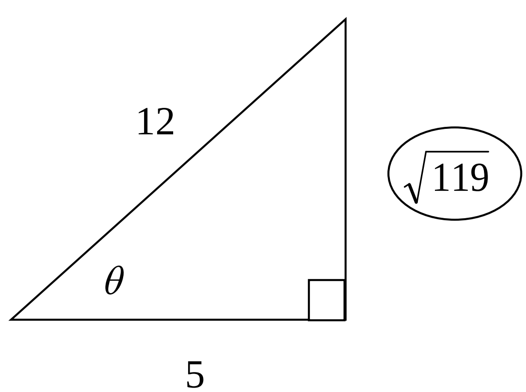
$$\cos\alpha = \frac{adj}{hyp} = \frac{1}{\sqrt{37}} \qquad \sec\alpha = \sqrt{37}$$

$$\sin\alpha = \frac{opp}{hyp} = \frac{6}{\sqrt{37}} \qquad \csc\alpha = \frac{\sqrt{37}}{6}$$

$$\tan\alpha = 6 \quad \text{(given)} \qquad \cot\alpha = \frac{1}{6}$$

3. $\sec\theta = \dfrac{12}{5}$ and θ is an acute angle

$$\sec\theta = \frac{12}{5} \Rightarrow \cos\theta = \frac{5}{12} = \frac{adj}{hyp}$$



NOTE: The number circled above was found using the Pythagorean Theorem by $\sqrt{144 - 25} = \sqrt{119}$.

$$\cos\theta = \frac{5}{12} \qquad\qquad \sec\theta = \frac{12}{5} \quad \text{(given)}$$

$$\sin\theta = \frac{opp}{hyp} = \frac{\sqrt{119}}{12} \qquad\qquad \csc\theta = \frac{12}{\sqrt{119}}$$

$$\tan\theta = \frac{opp}{adj} = \frac{\sqrt{119}}{5} \qquad\qquad \cot\theta = \frac{5}{\sqrt{119}}$$

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2. USING A RIGHT TRIANGLE TO FIND THE VALUE OF THE SIX TRIGONOMETRIC FUNCTIONS OF ANGLES IN THE FIRST, SECOND, THIRD, AND FOURTH QUADRANTS

<u>Examples</u> Determine what quadrant the following angles are in if given the following information.

1. $\cos\theta < 0$ and $\sin\theta > 0$

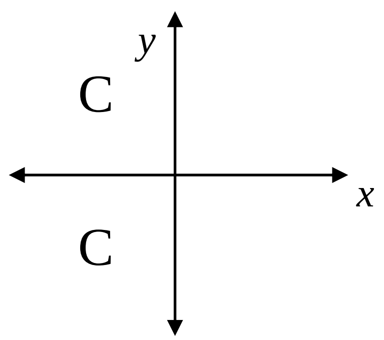
Method 1:

$\cos\theta < 0 \Rightarrow$ the x-coordinate of the point of intersection of the terminal side of the angle θ with the Unit Circle is negative. That is, $x < 0$.
$\sin\theta > 0 \Rightarrow$ the y-coordinate of the point of intersection of the terminal side of the angle θ with the Unit Circle is positive. That is, $y > 0$.

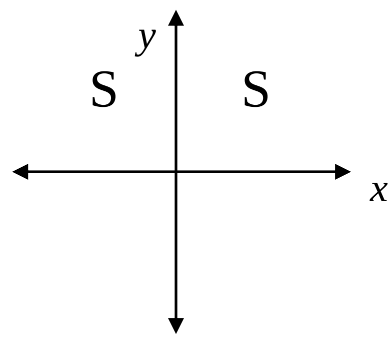
Thus, we have that $x < 0$ and $y > 0$. Thus, the angle θ is in the II quadrant.

Method 2:

$\cos\theta < 0$



$\sin\theta > 0$



Thus, the common quadrant is the II quadrant.

Method 3:

$\cos\theta < 0 \Rightarrow \theta$ is in either the II or III quadrant

$\sin\theta > 0 \Rightarrow \theta$ is in either the I or II quadrant

Thus, the common quadrant is the II quadrant.

Answer: II

2. $\tan\alpha < 0$ and $\cos\alpha > 0$

Method 1:

$\cos\alpha > 0 \Rightarrow$ the x-coordinate of the point of intersection of the terminal side of the angle α with the Unit Circle is positive. That is, $x > 0$.

The tangent of the angle α is the y-coordinate of the point of intersection of the terminal side of the angle α with the Unit Circle divided by the x-coordinate of the point of intersection. That is, $\tan\alpha = \dfrac{y}{x}$. Since we have that $\tan\alpha < 0$ and $x > 0$, then we have the following:

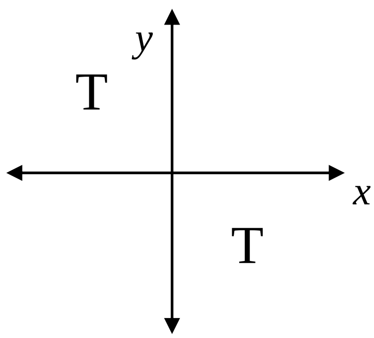
$$(-) = \tan\alpha = \frac{y}{x} = \frac{?}{(+)} \Rightarrow y < 0$$

since only a negative divided by a positive will result in a negative.

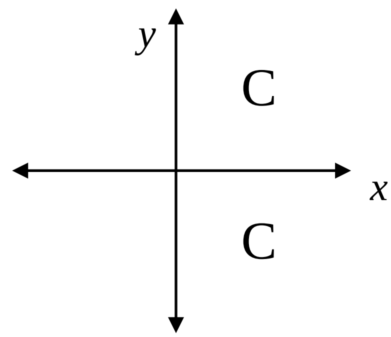
Thus, we have that $x > 0$ and $y < 0$. Thus, the angle α is in the IV quadrant.

Method 2:

$\tan\alpha < 0$



$\cos\alpha > 0$



Thus, the common quadrant is the IV quadrant.

Method 3:

$\tan\alpha < 0 \Rightarrow \alpha$ is in either the II or IV quadrant

$\cos\alpha > 0 \Rightarrow \alpha$ is in either the I or IV quadrant

Thus, the common quadrant is the IV quadrant.

Answer: IV

3. $\sin\beta < 0$ and $\cot\beta > 0$

Method 1:

$\sin\beta < 0 \Rightarrow y < 0$

$\cot \beta > 0 \Rightarrow \tan \beta > 0$. Since we have that $\tan \beta > 0$ and $y < 0$, then we have the following:

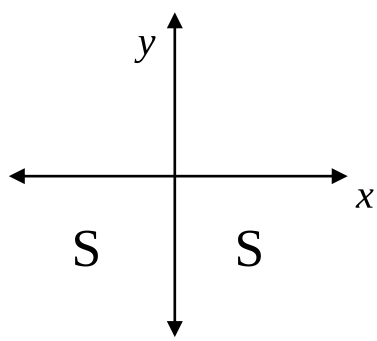
$$(+) = \tan \beta = \frac{y}{x} = \frac{(-)}{?} \Rightarrow x < 0$$

since only a negative divided by a negative will result in a positive.

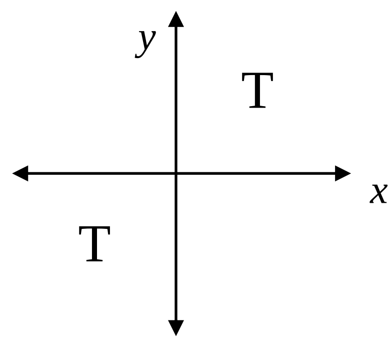
Thus, we have that $x < 0$ and $y < 0$. Thus, the angle β is in the III quadrant.

Method 2:

$\sin \beta < 0$



$\cot \beta > 0$ **or** $\tan \beta > 0$



Thus, the common quadrant is the III quadrant.

Method 3:

$\sin \beta < 0 \Rightarrow \beta$ is in either the III or IV quadrant

$\cot \beta > 0 \Rightarrow \beta$ is in either the I or III quadrant
or $\tan \beta > 0 \Rightarrow \beta$ is in either the I or III quadrant
Thus, the common quadrant is the III quadrant.

Answer: III

4. $\tan \gamma < 0$ and $\sec \gamma < 0$

Method 1:

$\sec \gamma < 0 \;\Rightarrow\; \cos \gamma < 0 \;\Rightarrow\; x < 0$

Since we have that $\tan \gamma < 0$ and $x < 0$, then we have the following:

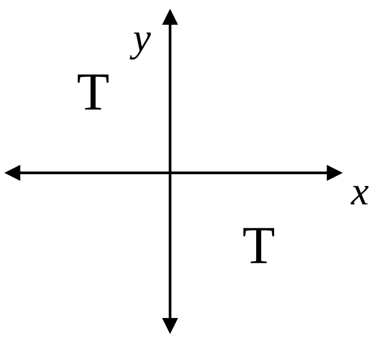
$$(-) = \tan \gamma = \frac{y}{x} = \frac{?}{(-)} \;\Rightarrow\; y > 0$$

since only a positive divided by a negative will result in a negative.

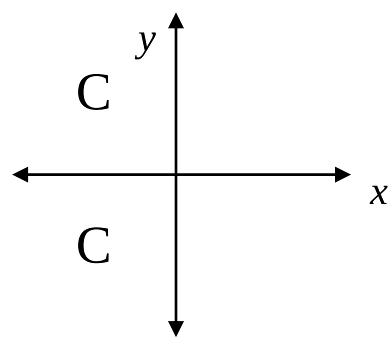
Thus, we have that $x < 0$ and $y > 0$. Thus, the angle γ is in the II quadrant.

Method 2:

$\tan \gamma < 0$



$\sec \gamma < 0$ **or** $\cos \gamma < 0$



Thus, the common quadrant is the II quadrant.

Method 3:

$\tan \gamma < 0 \;\Rightarrow\; \gamma$ is in either the II or IV quadrant

$\sec \gamma < 0 \;\Rightarrow\; \gamma$ is in either the II or III quadrant
or $\cos \gamma < 0 \;\Rightarrow\; \gamma$ is in either the II or III quadrant

Thus, the common quadrant is the II quadrant.

Answer: II

5. $\csc\theta < 0$ and $\sec\theta > 0$

Method 1:

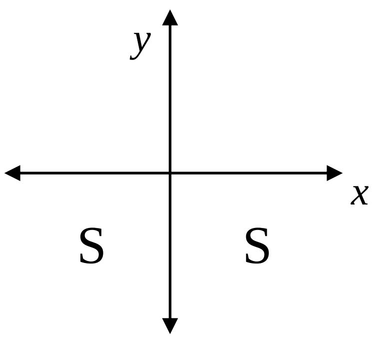
$\csc\theta < 0 \Rightarrow \sin\theta < 0 \Rightarrow y < 0$

$\sec\theta > 0 \Rightarrow \cos\theta > 0 \Rightarrow x > 0$

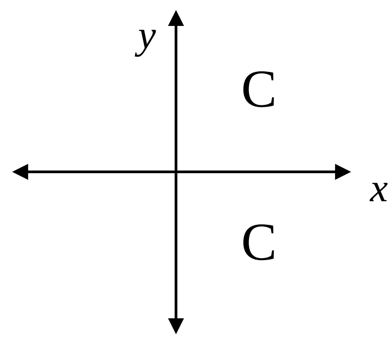
Thus, we have that $x > 0$ and $y < 0$. Thus, the angle θ is in the IV quadrant.

Method 2:

$\csc\theta < 0$ **or** $\sin\theta < 0$



$\sec\theta > 0$ **or** $\cos\theta > 0$



Thus, the common quadrant is the IV quadrant.

Method 3:

$\csc\theta < 0 \Rightarrow \theta$ is in either the III or IV quadrant
or $\sin\theta < 0 \Rightarrow \theta$ is in either the III or IV quadrant

$\sec\theta > 0 \Rightarrow \theta$ is in either the I or IV quadrant
or $\cos\theta > 0 \Rightarrow \theta$ is in either the I or IV quadrant

Thus, the common quadrant is the IV quadrant.

Answer: IV

Examples Use a right triangle to find the exact value of the other five trigonometric functions if given the following.

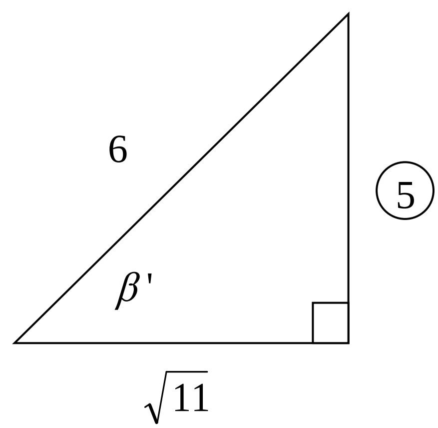
1. $\cos\beta = \dfrac{\sqrt{11}}{6}$ and β is in the IV quadrant

Since the angle β is in the IV quadrant, then it is not an acute angle. Thus, the angle β can not be put in a right triangle. Thus, we have that

$$\cos\beta = \frac{\sqrt{11}}{6} \neq \frac{adj}{hyp}$$

However, the reference angle β' of the angle β is an acute and can be put in a triangle. Thus, we have that

$$\cos\beta = \frac{\sqrt{11}}{6} \Rightarrow \cos\beta' = \frac{\sqrt{11}}{6} = \frac{adj}{hyp}$$



NOTE: The number circled above was found using the Pythagorean Theorem by $\sqrt{36-11} = \sqrt{25} = 5$.

From the triangle, we obtain that

$\cos\beta' = \dfrac{\sqrt{11}}{6}$ ("given") $\qquad \sin\beta' = \dfrac{opp}{hyp} = \dfrac{5}{6} \qquad \tan\beta' = \dfrac{opp}{adj} = \dfrac{5}{\sqrt{11}}$

Since the angle β is in the IV quadrant, then we know that the sine (and cosecant) of β is negative and the tangent (and cotangent) of β is negative. Now, using reference angles, we obtain that

$$\cos\beta = \frac{\sqrt{11}}{6} \quad \text{(given)} \qquad \sec\beta = \frac{6}{\sqrt{11}}$$

$$\sin\beta = -\sin\beta' = -\frac{5}{6} \qquad \csc\beta = -\frac{6}{5}$$

$$\tan\beta = -\tan\beta' = -\frac{5}{\sqrt{11}} \qquad \cot\beta = -\frac{\sqrt{11}}{5}$$

2. $\csc\alpha = -\dfrac{17}{8}$ and $\cot\alpha > 0$

First, we will use Method 1 from above in order to determine what quadrant the terminal side of the angle α is in.

$\csc\alpha < 0 \Rightarrow \sin\alpha < 0 \Rightarrow y < 0$

$\cot\alpha > 0 \Rightarrow \tan\alpha > 0$. Since we have that $\tan\alpha > 0$ and $y < 0$, then we have the following:

$$(+) = \tan\alpha = \frac{y}{x} = \frac{(-)}{?} \Rightarrow x < 0$$

since only a negative divided by a negative will result in a positive.

Thus, we have that $x < 0$ and $y < 0$. Thus, the angle α is in the III quadrant.

You may use Method 2 or Method 3 from above if you prefer.

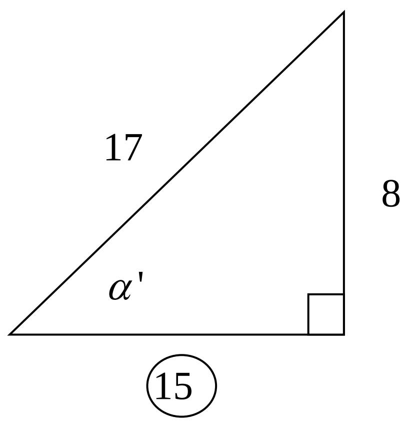
$$\csc\alpha = -\frac{17}{8} \Rightarrow \sin\alpha = -\frac{8}{17}$$

Since the angle α is in the III quadrant, then it is not an acute angle. Thus, the angle α can not be put in a right triangle. Thus, we have that

$$\sin\alpha = -\frac{8}{17} \neq \frac{opp}{hyp}$$

However, the reference angle α' of the angle α is an acute and can be put in a triangle. Thus, we have that

$$\sin\alpha = -\frac{8}{17} \Rightarrow \sin\alpha' = \frac{8}{17} = \frac{opp}{hyp}$$



NOTE: The number circled above was found using the Pythagorean Theorem by $\sqrt{289 - 64} = \sqrt{225} = 15$.

From the triangle, we obtain that

$$\cos\alpha' = \frac{adj}{hyp} = \frac{15}{17} \qquad \sin\alpha' = \frac{8}{17} \text{ (“given”)} \qquad \tan\alpha' = \frac{opp}{adj} = \frac{8}{15}$$

Since the angle α is in the III quadrant, then we know that the cosine (and secant) of α is negative and the tangent (and cotangent) of α is positive. Now, using reference angles, we obtain that

$$\cos\alpha = -\cos\alpha' = -\frac{15}{17} \qquad \sec\alpha = -\frac{17}{15}$$

$\sin \alpha = -\frac{8}{17}$ $\qquad$ $\csc \alpha = -\frac{17}{8}$ (given)

$\tan \alpha = \tan \alpha' = \frac{8}{15}$ $\qquad$ $\cot \alpha = \frac{15}{8}$

3. $\tan \theta = -\frac{5}{7}$ and $\sec \theta < 0$

First, we will use Method 1 from above in order to determine what quadrant the terminal side of the angle θ is in.

$\sec \theta < 0 \Rightarrow \cos \theta < 0 \Rightarrow x < 0$

Since we have that $\tan \theta < 0$ and $x < 0$, then we have the following:

$$(-) = \tan \theta = \frac{y}{x} = \frac{?}{(-)} \Rightarrow y > 0$$

since only a positive divided by a negative will result in a negative.

Thus, we have that $x < 0$ and $y > 0$. Thus, the angle θ is in the II quadrant.

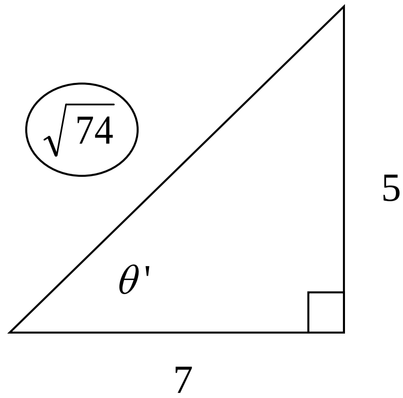
You may use Method 2 or Method 3 from above if you prefer.

Since the angle θ is in the II quadrant, then it is not an acute angle. Thus, the angle θ can not be put in a right triangle. Thus, we have that

$$\tan \theta = -\frac{5}{7} \neq \frac{opp}{adj}$$

However, the reference angle θ' of the angle θ is an acute and can be put in a triangle. Thus, we have that

$$\tan \theta = -\frac{5}{7} \Rightarrow \tan \theta' = \frac{5}{7} = \frac{opp}{adj}$$



NOTE: The number circled above was found using the Pythagorean Theorem by $\sqrt{49 + 25} = \sqrt{74}$.

From the triangle, we obtain that

$$\cos\theta' = \frac{adj}{hyp} = \frac{7}{\sqrt{74}} \qquad \sin\theta' = \frac{opp}{hyp} = \frac{5}{\sqrt{74}} \qquad \tan\theta' = \frac{5}{7} \text{ (“given”)}$$

Since the angle θ is in the II quadrant, then we know that the cosine (and secant) of θ is negative and the sine (and cosecant) of θ is positive. Now, using reference angles, we obtain that

$$\cos\theta = -\cos\theta' = -\frac{7}{\sqrt{74}} \qquad \sec\theta = -\frac{\sqrt{74}}{7}$$

$$\sin\theta = \sin\theta' = \frac{5}{\sqrt{74}} \qquad \csc\theta = \frac{\sqrt{74}}{5}$$

$$\tan\theta = -\frac{5}{7} \text{ (given)} \qquad \cot\theta = -\frac{7}{5}$$

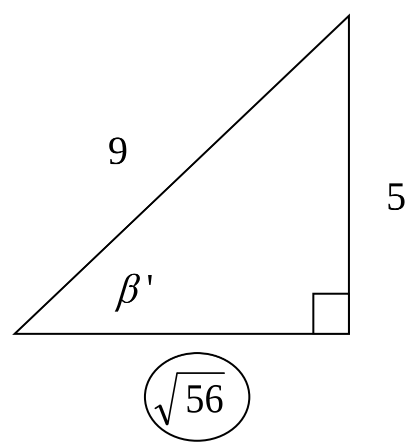
4. $\sin\beta = \dfrac{5}{9}$ and β is in the II quadrant

Since the angle β is in the II quadrant, then it is not an acute angle. Thus, the angle β can not be put in a right triangle. Thus, we have that

$$\sin \beta = \frac{5}{9} \neq \frac{opp}{hyp}$$

However, the reference angle β' of the angle β is an acute and can be put in a triangle. Thus, we have that

$$\sin \beta = \frac{5}{9} \Rightarrow \sin \beta' = \frac{5}{9} = \frac{opp}{hyp}$$



NOTE: The number circled above was found using the Pythagorean Theorem by $\sqrt{81 - 25} = \sqrt{56}$.

From the triangle, we obtain that

$$\cos \beta' = \frac{adj}{hyp} = \frac{\sqrt{56}}{9} \qquad \sin \beta' = \frac{5}{9} \text{ ("given")} \qquad \tan \beta' = \frac{opp}{adj} = \frac{5}{\sqrt{56}}$$

Since the angle β is in the II quadrant, then we know that the cosine (and secant) of β is negative and the tangent (and cotangent) of β is negative. Now, using reference angles, we obtain that

$$\cos\beta = -\cos\beta' = -\frac{\sqrt{56}}{9} \qquad \sec\beta = -\frac{9}{\sqrt{56}}$$

$$\sin\beta = \frac{5}{9} \text{ (given)} \qquad \csc\beta = \frac{9}{5}$$

$$\tan\beta = -\tan\beta' = -\frac{5}{\sqrt{56}} \qquad \cot\beta = -\frac{\sqrt{56}}{5}$$

5. $\sec\alpha = -5$ and $\sin\alpha < 0$

First, we will use Method 1 from above in order to determine what quadrant the terminal side of the angle α is in.

$\sec\alpha < 0 \Rightarrow \cos\alpha < 0 \Rightarrow x < 0$

$\sin\alpha < 0 \Rightarrow y < 0$

Thus, we have that $x < 0$ and $y < 0$. Thus, the angle α is in the III quadrant.

You may use Method 2 or Method 3 from above if you prefer.

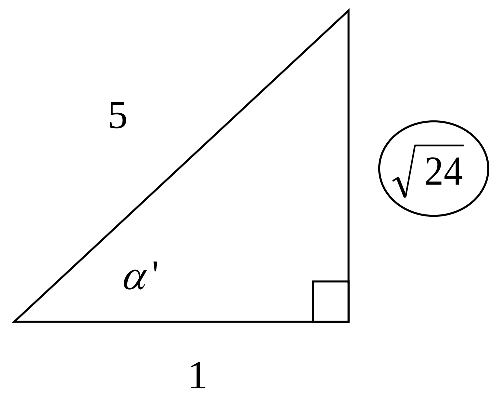
$$\sec\alpha = -5 \Rightarrow \cos\alpha = -\frac{1}{5}$$

Since the angle α is in the III quadrant, then it is not an acute angle. Thus, the angle α can not be put in a right triangle. Thus, we have that

$$\cos\alpha = -\frac{1}{5} \neq \frac{adj}{hyp}$$

However, the reference angle α' of the angle α is an acute and can be put in a triangle. Thus, we have that

$$\cos\alpha = -\frac{1}{5} \Rightarrow \cos\alpha' = \frac{1}{5} = \frac{adj}{hyp}$$



NOTE: The number circled above was found using the Pythagorean Theorem by $\sqrt{25-1} = \sqrt{24}$.

From the triangle, we obtain that

$\cos\alpha' = \frac{1}{5}$ ("given") $\qquad \sin\alpha' = \frac{opp}{hyp} = \frac{\sqrt{24}}{5} \qquad \tan\alpha' = \frac{opp}{adj} = \sqrt{24}$

Since the angle α is in the III quadrant, then we know that the sine (and cosecant) of α is negative and the tangent (and cotangent) of α is positive. Now, using reference angles, we obtain that

$$\cos\alpha = -\frac{1}{5} \qquad \sec\alpha = -5 \quad \text{(given)}$$

$$\sin\alpha = -\sin\alpha' = -\frac{\sqrt{24}}{5} \qquad \csc\alpha = -\frac{5}{\sqrt{24}}$$

$$\tan\alpha = \tan\alpha' = \sqrt{24} \qquad \cot\alpha = \frac{1}{\sqrt{24}}$$

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