LESSON 8 THE GRAPHS OF THE TRIGONOMETRIC FUNCTIONS

Topics in this lesson:

1. SINE GRAPHS

Example Use the Unit Circle to graph two cycles of the function $y = \sin x$ on the interval $[0, 4\pi]$.

Example Use the Unit Circle to graph two cycles of the function $y = \sin x$ on the interval $[-4\pi, 0]$.

Definition The amplitude of a trigonometric function is one-half of the difference between the maximum value of the function and the minimum value of the function if the function has both of these values.

NOTE: The maximum value of a function is the largest y-coordinate on the graph of the function and the minimum value of a function is the smallest y-coordinate on the graph of the function if the graph has both of these values.

The sine and cosine functions will have an amplitude. However, the tangent, cotangent, secant, and cosecant functions do not have an amplitude because these functions do not have a maximum value nor a minimum value.

Definition The period of a trigonometric function is the distance needed to complete one cycle of the graph of the function.

All the trigonometric functions have a period.

For the function $y = \sin x$, the amplitude of the function is 1 and the period is 2π.

Given the function $y = a \sin bx$, the amplitude of this function is $|a|$ and the period is $\dfrac{2\pi}{|b|}$.

Theorem The sine function is an odd function. That is, $\sin(-\theta) = -\sin\theta$ for all θ in the domain of the function.

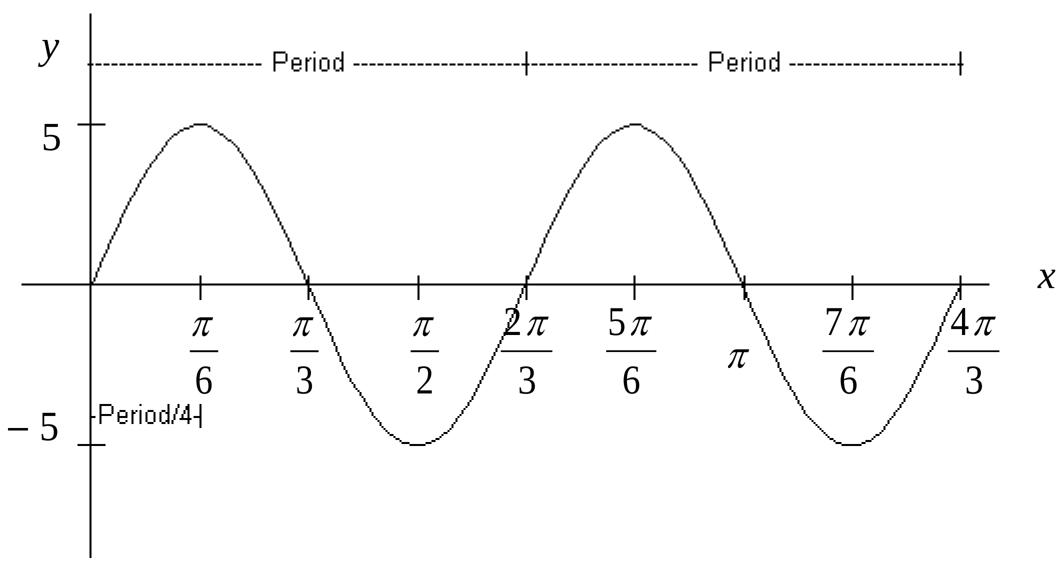
NOTE: The domain of the sine function is all real numbers.

Examples Sketch two cycles of the graph of the following functions. Label the numbers on the *x*- and *y*-axes.

1. $y = 5 \sin 3x$

Amplitude $= |5| = 5$ Period $= \dfrac{2\pi}{|3|} = \dfrac{2\pi}{3}$

$$\frac{1}{4} \cdot \text{period} = \frac{1}{4} \cdot \frac{2\pi}{3} = \frac{1}{2} \cdot \frac{\pi}{3} = \frac{\pi}{6}$$



NOTE: The first cycle begins at 0. We do not need to label that number. Since the period is $\dfrac{2\pi}{3}$, the first cycle ends at $\dfrac{2\pi}{3}$, which is obtained by

$0 + \frac{2\pi}{3} = \frac{2\pi}{3}$. That is, we add the period of $\frac{2\pi}{3}$ to the starting point of 0. The second cycle ends at $\frac{4\pi}{3}$, which is obtained by $\frac{2\pi}{3} + \frac{2\pi}{3} = \frac{4\pi}{3}$. That is, we add the period of $\frac{2\pi}{3}$ to the starting point of $\frac{2\pi}{3}$.

Now, the rest of the numbers on the *x*-axis were obtained in the following manner:

The $\frac{\pi}{6}$ was obtained by $0 + \frac{\pi}{6} = \frac{\pi}{6}$. That is, we add $\frac{\pi}{6}$, which is one-fourth of the period, to the starting point of 0.

The $\frac{\pi}{3}$ was obtained by $\frac{\pi}{6} + \frac{\pi}{6} = \frac{2\pi}{6} = \frac{\pi}{3}$. That is, we add $\frac{\pi}{6}$, which is one-fourth of the period, to the next starting point of $\frac{\pi}{6}$.

The $\frac{\pi}{2}$ was obtained by $\frac{2\pi}{6} + \frac{\pi}{6} = \frac{3\pi}{6} = \frac{\pi}{2}$. That is, we add $\frac{\pi}{6}$, which is one-fourth of the period, to the next starting point of $\frac{2\pi}{6}$.

We can check the $\frac{2\pi}{3}$ by $\frac{3\pi}{6} + \frac{\pi}{6} = \frac{4\pi}{6} = \frac{2\pi}{3}$. That is, we add $\frac{\pi}{6}$, which is one-fourth of the period, to the next starting point of $\frac{3\pi}{6}$.

The $\frac{5\pi}{6}$ was obtained by $\frac{4\pi}{6} + \frac{\pi}{6} = \frac{5\pi}{6}$. That is, we add $\frac{\pi}{6}$, which is one-fourth of the period, to the next starting point of $\frac{4\pi}{6}$. Or, you can obtain the

$\frac{5\pi}{6}$ by adding the period of $\frac{2\pi}{3}$ to the previous $\frac{\pi}{6}$ in the first cycle. Thus, $\frac{\pi}{6} + \frac{2\pi}{3} = \frac{\pi}{6} + \frac{4\pi}{6} = \frac{5\pi}{6}$.

The π was obtained by $\frac{5\pi}{6} + \frac{\pi}{6} = \frac{6\pi}{6} = \pi$. That is, we add $\frac{\pi}{6}$, which is one-fourth of the period, to the next starting point of $\frac{5\pi}{6}$. Or, you can obtain the π by adding the period of $\frac{2\pi}{3}$ to the previous $\frac{\pi}{3}$ in the first cycle. Thus, $\frac{\pi}{3} + \frac{2\pi}{3} = \frac{3\pi}{3} = \pi$.

The $\frac{7\pi}{6}$ was obtained by $\frac{6\pi}{6} + \frac{\pi}{6} = \frac{7\pi}{6}$. That is, we add $\frac{\pi}{6}$, which is one-fourth of the period, to the next starting point of $\frac{6\pi}{6}$. Or, you can obtain the $\frac{7\pi}{6}$ by adding the period of $\frac{2\pi}{3}$ to the previous $\frac{\pi}{2}$ in the first cycle. Thus, $\frac{\pi}{2} + \frac{2\pi}{3} = \frac{3\pi}{6} + \frac{4\pi}{6} = \frac{7\pi}{6}$.

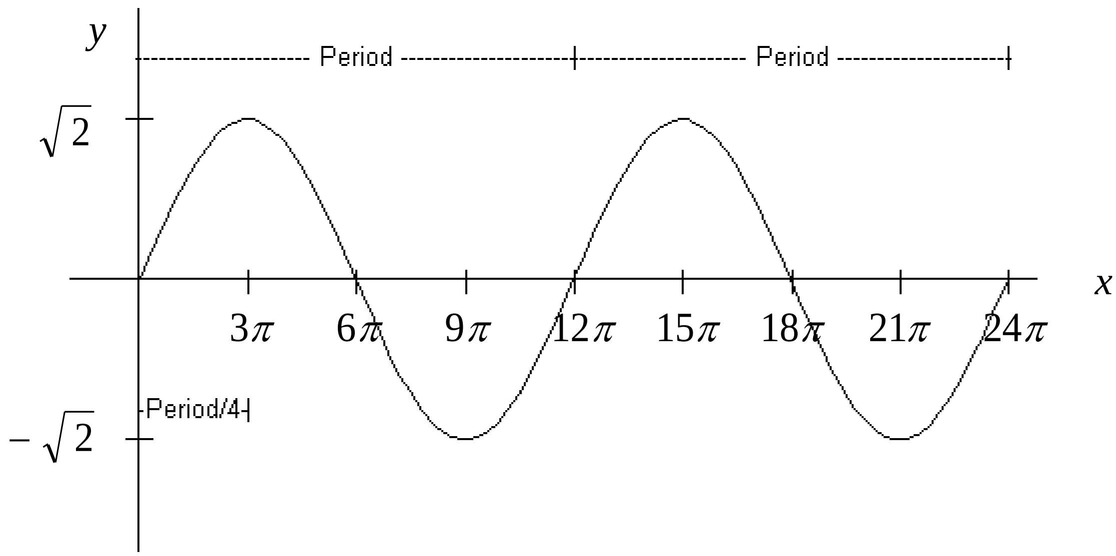
We can check the $\frac{4\pi}{3}$ by $\frac{7\pi}{6} + \frac{\pi}{6} = \frac{8\pi}{6} = \frac{4\pi}{3}$. That is, we add $\frac{\pi}{6}$, which is one-fourth, of the period to the next starting point of $\frac{7\pi}{6}$.

The graph of two cycles of $y = 5\sin 3x$ in blue compared with the graph of two cycles of $y = \sin x$ in red.

2. $y = \sqrt{2}\sin\left(\frac{x}{6}\right)$

Amplitude $= \sqrt{2}$ Period $= \dfrac{2\pi}{\frac{1}{6}} = 2\pi \cdot 6 = 12\pi$

$\frac{1}{4} \cdot \text{period} = \frac{1}{4} \cdot 12\pi = \frac{12\pi}{4} = 3\pi$



NOTE: The first cycle begins at 0. We do not need to label that number. Since the period is 12π, the first cycle ends at 12π, which is obtained by $0 + 12\pi = 12\pi$. That is, we add the period of 12π to the starting point of 0. The second cycle ends at 24π, which is obtained by $12\pi + 12\pi = 24\pi$. That is, we add the period of 12π to the starting point of 12π.

Now, the rest of the numbers on the *x*-axis were obtained in the following manner:

The 3π was obtained by $0 + 3\pi = 3\pi$. That is, we add 3π, which is one-fourth of the period, to the starting point of 0.

The 6π was obtained by $3\pi + 3\pi = 6\pi$. That is, we add 3π, which is one-fourth of the period, to the next starting point of 3π.

The 9π was obtained by $6\pi + 3\pi = 9\pi$. That is, we add 3π, which is one-fourth of the period, to the next starting point of 6π.

We can check the 12π by $9\pi + 3\pi = 12\pi$. That is, we add 3π, which is one-fourth of the period, to the next starting point of 9π.

The 15π was obtained by $12\pi + 3\pi = 15\pi$. That is, we add 3π, which is one-fourth of the period, to the next starting point of 12π. Or, you can obtain the 15π by adding the period of 12π to the previous 3π in the first cycle. Thus, $3\pi + 12\pi = 15\pi$.

The 18π was obtained by $15\pi + 3\pi = 18\pi$. That is, we add 3π, which is one-fourth of the period, to the next starting point of 15π. Or, you can obtain the 18π by adding the period of 12π to the previous 6π in the first cycle. Thus, $6\pi + 12\pi = 18\pi$.

The 21π was obtained by $18\pi + 3\pi = 21\pi$. That is, we add 3π, which is one-fourth of the period, to the next starting point of 18π. Or, you can obtain the 21π by adding the period of 12π to the previous 9π in the first cycle. Thus, $9\pi + 12\pi = 21\pi$.

We can check the 24π by $21\pi + 3\pi = 24\pi$. That is, we add 3π, which is one-fourth, of the period to the next starting point of 21π.

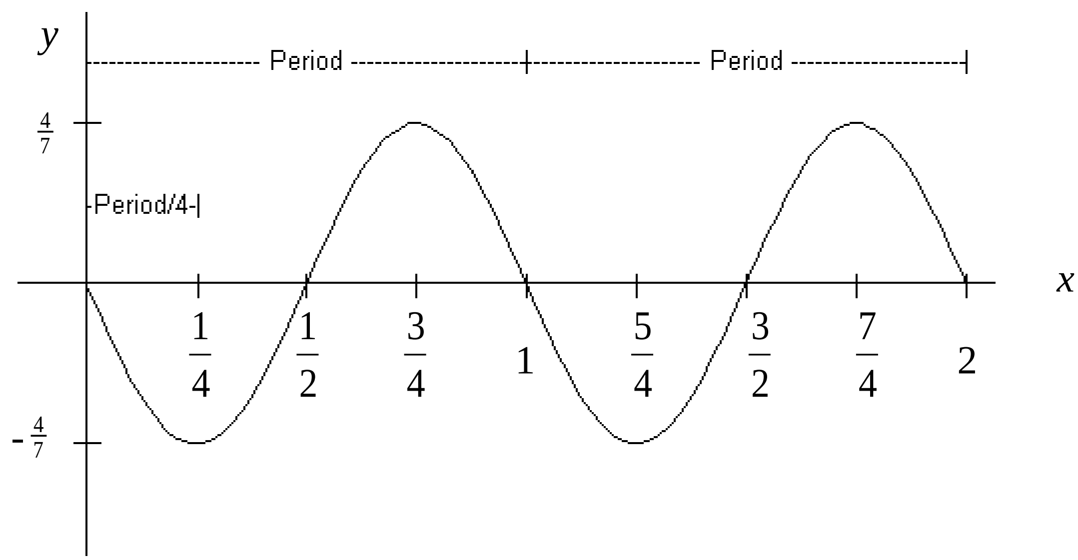
The graph of two cycles of $y = \sqrt{2}\sin\left(\frac{x}{6}\right)$ in blue compared with the graph of two cycles of $y = \sin x$ in red.

3. $y = -\frac{4}{7}\sin 2\pi x$

NOTE: Since the sine function is being multiplied by a **negative** $\frac{4}{7}$, then the graph will be inverted. Thus, we will need to draw two inverted sine cycles.

Amplitude $= \left| -\frac{4}{7} \right| = \frac{4}{7}$ Period $= \frac{2\pi}{2\pi} = 1$

$\frac{1}{4} \cdot$ period $= \frac{1}{4} \cdot 1 = \frac{1}{4}$



NOTE: The first cycle begins at 0. We do not need to label that number. Since the period is 1, the first cycle ends at 1, which is obtained by $0 + 1 = 1$. That is, we add the period of 1 to the starting point of 0. The second cycle ends at 2, which is obtained by $1 + 1 = 2$. That is, we add the period of 1 to the starting point of 1.

Now, the rest of the numbers on the *x*-axis were obtained in the following manner:

The $\frac{1}{4}$ was obtained by $0 + \frac{1}{4} = \frac{1}{4}$. That is, we add $\frac{1}{4}$, which is one-fourth of the period, to the starting point of 0.

The $\frac{1}{2}$ was obtained by $\frac{1}{4} + \frac{1}{4} = \frac{2}{4} = \frac{1}{2}$. That is, we add $\frac{1}{4}$, which is one-fourth of the period, to the next starting point of $\frac{1}{4}$.

The $\frac{3}{4}$ was obtained by $\frac{2}{4} + \frac{1}{4} = \frac{3}{4}$. That is, we add $\frac{1}{4}$, which is one-fourth of the period, to the next starting point of $\frac{2}{4}$.

We can check the 1 by $\frac{3}{4} + \frac{1}{4} = \frac{4}{4} = 1$. That is, we add $\frac{1}{4}$, which is one-fourth of the period, to the next starting point of $\frac{3}{4}$.

The $\frac{5}{4}$ was obtained by $\frac{4}{4} + \frac{1}{4} = \frac{5}{4}$. That is, we add $\frac{1}{4}$, which is one-fourth of the period, to the next starting point of $\frac{4}{4}$. Or, you can obtain the $\frac{5}{4}$ by adding the period of 1 to the previous $\frac{1}{4}$ in the first cycle. Thus, $\frac{1}{4} + 1 = \frac{1}{4} + \frac{4}{4} = \frac{5}{4}$.

The $\frac{3}{2}$ was obtained by $\frac{5}{4} + \frac{1}{4} = \frac{6}{4} = \frac{3}{2}$. That is, we add $\frac{1}{4}$, which is one-fourth of the period, to the next starting point of $\frac{5}{4}$. Or, you can obtain the $\frac{3}{2}$ by adding the period of 1 to the previous $\frac{1}{2}$ in the first cycle. Thus, $\frac{1}{2} + 1 = \frac{1}{2} + \frac{2}{2} = \frac{3}{2}$.

The $\frac{7}{4}$ was obtained by $\frac{6}{4} + \frac{1}{4} = \frac{7}{4}$. That is, we add $\frac{1}{4}$, which is one-fourth of the period, to the next starting point of $\frac{6}{4}$. Or, you can obtain the $\frac{7}{4}$ by adding the period of 1 to the previous $\frac{3}{4}$ in the first cycle. Thus, $\frac{3}{4} + 1 = \frac{3}{4} + \frac{4}{4} = \frac{7}{4}$.

We can check the 2 by $\frac{7}{4} + \frac{1}{4} = \frac{8}{4} = 2$. That is, we add $\frac{1}{4}$, which is one-fourth, of the period to the next starting point of $\frac{7}{4}$.

The graph of two cycles of $y = -\frac{4}{7}\sin 2\pi x$ in blue compared with the graph of two cycles of $y = -\sin x$ in red.

4. $y = 8\sin\left(-\frac{7x}{3}\right)$

NOTE: Since the sine function is an odd function, then $\sin\left(-\frac{7x}{3}\right) = -\sin\left(\frac{7}{3}x\right)$. Thus, we have that

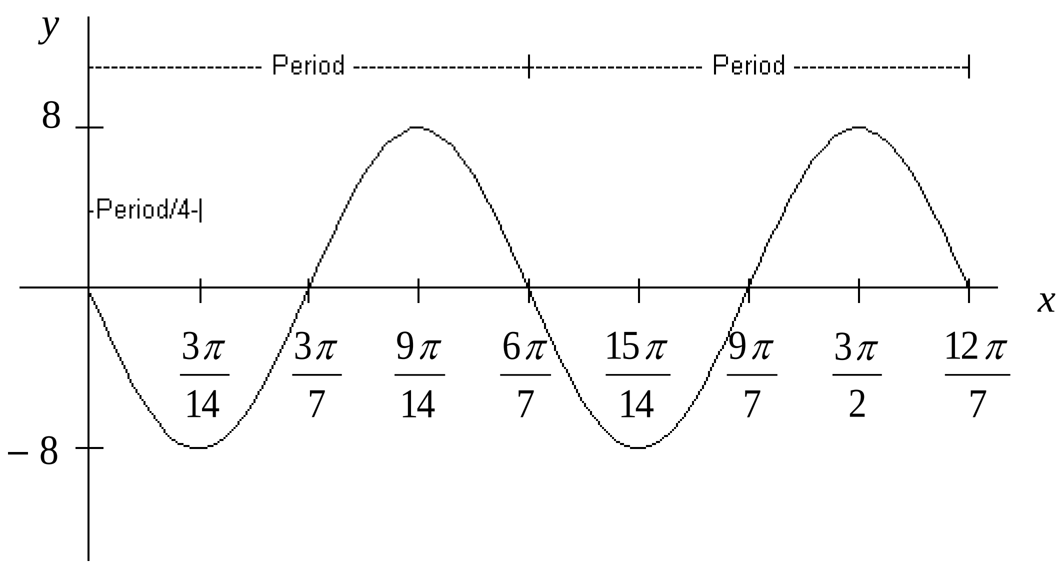
$$y = 8\sin\left(-\frac{7x}{3}\right) = -8\sin\left(\frac{7}{3}x\right)$$

Since the sine function is being multiplied by a **negative** 8, then the graph will be inverted. Thus, we will need to draw two inverted sine cycles.

Amplitude = 8

$$\text{Period} = \frac{2\pi}{\frac{7}{3}} = 2\pi \cdot \frac{3}{7} = \frac{6\pi}{7}$$

$$\frac{1}{4} \cdot \text{period} = \frac{1}{4} \cdot \frac{6\pi}{7} = \frac{1}{2} \cdot \frac{3\pi}{7} = \frac{3\pi}{14}$$



NOTE: The first cycle begins at 0. We do not need to label that number. Since the period is $\frac{6\pi}{7}$, the first cycle ends at $\frac{6\pi}{7}$, which is obtained by

$0 + \frac{6\pi}{7} = \frac{6\pi}{7}$. That is, we add the period of $\frac{6\pi}{7}$ to the starting point of 0. The second cycle ends at $\frac{12\pi}{7}$, which is obtained by $\frac{6\pi}{7} + \frac{6\pi}{7} = \frac{12\pi}{7}$. That is, we add the period of $\frac{6\pi}{7}$ to the starting point of $\frac{6\pi}{7}$.

Now, the rest of the numbers on the x-axis were obtained in the following manner:

The $\frac{3\pi}{14}$ was obtained by $0 + \frac{3\pi}{14} = \frac{3\pi}{14}$. That is, we add $\frac{3\pi}{14}$, which is one-fourth of the period, to the starting point of 0.

The $\frac{3\pi}{7}$ was obtained by $\frac{3\pi}{14} + \frac{3\pi}{14} = \frac{6\pi}{14} = \frac{3\pi}{7}$. That is, we add $\frac{3\pi}{14}$, which is one-fourth of the period, to the next starting point of $\frac{3\pi}{14}$.

The $\frac{9\pi}{14}$ was obtained by $\frac{6\pi}{14} + \frac{3\pi}{14} = \frac{9\pi}{14}$. That is, we add $\frac{3\pi}{14}$, which is one-fourth of the period, to the next starting point of $\frac{6\pi}{14}$.

We can check the $\frac{6\pi}{7}$ by $\frac{9\pi}{14} + \frac{3\pi}{14} = \frac{12\pi}{14} = \frac{6\pi}{7}$. That is, we add $\frac{3\pi}{14}$, which is one-fourth of the period, to the next starting point of $\frac{9\pi}{14}$.

The $\frac{15\pi}{14}$ was obtained by $\frac{12\pi}{14} + \frac{3\pi}{14} = \frac{15\pi}{14}$. That is, we add $\frac{3\pi}{14}$, which is one-fourth of the period, to the next starting point of $\frac{12\pi}{14}$. Or, you can obtain the $\frac{15\pi}{14}$ by adding the period of $\frac{6\pi}{7}$ to the previous $\frac{3\pi}{14}$ in the first cycle. Thus, $\frac{3\pi}{14} + \frac{6\pi}{7} = \frac{3\pi}{14} + \frac{12\pi}{14} = \frac{15\pi}{14}$.

The $\frac{9\pi}{7}$ was obtained by $\frac{15\pi}{14} + \frac{3\pi}{14} = \frac{18\pi}{14} = \frac{9\pi}{7}$. That is, we add $\frac{3\pi}{14}$, which is one-fourth of the period, to the next starting point of $\frac{15\pi}{14}$. Or, you can obtain the $\frac{9\pi}{7}$ by adding the period of $\frac{6\pi}{7}$ to the previous $\frac{3\pi}{7}$ in the first cycle. Thus, $\frac{3\pi}{7} + \frac{6\pi}{7} = \frac{9\pi}{7}$.
The $\frac{3\pi}{2}$ was obtained by $\frac{18\pi}{14} + \frac{3\pi}{14} = \frac{21\pi}{14} = \frac{3\pi}{2}$. That is, we add $\frac{3\pi}{14}$, which is one-fourth of the period, to the next starting point of $\frac{18\pi}{14}$. Or, you can obtain the $\frac{3\pi}{2}$ by adding the period of $\frac{6\pi}{7}$ to the previous $\frac{9\pi}{14}$ in the first cycle. Thus, $\frac{9\pi}{14} + \frac{6\pi}{7} = \frac{9\pi}{14} + \frac{12\pi}{14} = \frac{21\pi}{14} = \frac{3\pi}{2}$.

We can check the $\frac{12\pi}{7}$ by $\frac{21\pi}{14} + \frac{3\pi}{14} = \frac{24\pi}{14} = \frac{12\pi}{7}$. That is, we add $\frac{3\pi}{14}$, which is one-fourth, of the period to the next starting point of $\frac{21\pi}{14}$.

The graph of two cycles of $y = -\frac{4}{7}\sin 2\pi x$ in blue compared with the graph of two cycles of $y = -\sin x$ in red.

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2. COSINE GRAPHS

Example Use the Unit Circle to graph two cycles of the function $y = \cos x$ on the interval $[0, 4\pi]$.

Example Use the Unit Circle to graph two cycles of the function $y = \cos x$ on the interval $[-4\pi, 0]$.

For the function $y = \cos x$, the amplitude of the function is 1 and the period is 2π.

Given the function $y = a\cos bx$, the amplitude of this function is $|a|$ and the period is $\frac{2\pi}{|b|}$.

Theorem The cosine function is an even function. That is, $\cos(-\theta) = \cos\theta$ for all θ in the domain of the function.

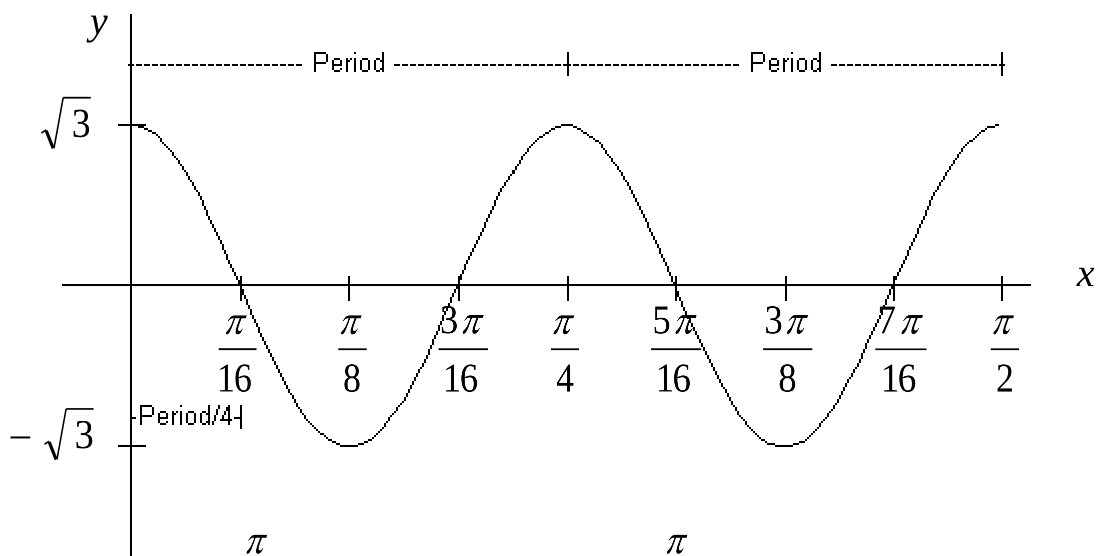
NOTE: The domain of the cosine function is all real numbers.

Examples Sketch two cycles of the graph of the following functions.

1. $y = \sqrt{3}\cos 8x$

Amplitude $= |\sqrt{3}| = \sqrt{3}$ Period $= \frac{2\pi}{|8|} = \frac{2\pi}{8} = \frac{\pi}{4}$

$$\frac{1}{4}\cdot \text{period} = \frac{1}{4}\cdot\frac{\pi}{4} = \frac{\pi}{16}$$



Since the period is $\frac{\pi}{4}$, the first cycle ends at $\frac{\pi}{4}$ and the second cycle ends at $\frac{2\pi}{4} = \frac{\pi}{2}$. The other numbers on the *x*-axis were obtained by the following:

$$0 + \frac{\pi}{16} = \frac{\pi}{16}$$

$$\frac{\pi}{16} + \frac{\pi}{16} = \frac{2\pi}{16} = \frac{\pi}{8}$$

$$\frac{2\pi}{16} + \frac{\pi}{16} = \frac{3\pi}{16}$$

Check: $\frac{3\pi}{16} + \frac{\pi}{16} = \frac{4\pi}{16} = \frac{\pi}{4}$

$\frac{4\pi}{16} + \frac{\pi}{16} = \frac{5\pi}{16}$ **OR** $\frac{\pi}{16} + \frac{\pi}{4} = \frac{\pi}{16} + \frac{4\pi}{16} = \frac{5\pi}{16}$

$\frac{5\pi}{16} + \frac{\pi}{16} = \frac{6\pi}{16} = \frac{3\pi}{8}$ **OR** $\frac{\pi}{8} + \frac{\pi}{4} = \frac{\pi}{8} + \frac{2\pi}{8} = \frac{3\pi}{8}$

$$\frac{6\pi}{16} + \frac{\pi}{16} = \frac{7\pi}{16} \quad \textbf{OR} \quad \frac{3\pi}{16} + \frac{\pi}{4} = \frac{3\pi}{16} + \frac{4\pi}{16} = \frac{7\pi}{16}$$

Check: $\dfrac{7\pi}{16} + \dfrac{\pi}{16} = \dfrac{8\pi}{16} = \dfrac{\pi}{2}$

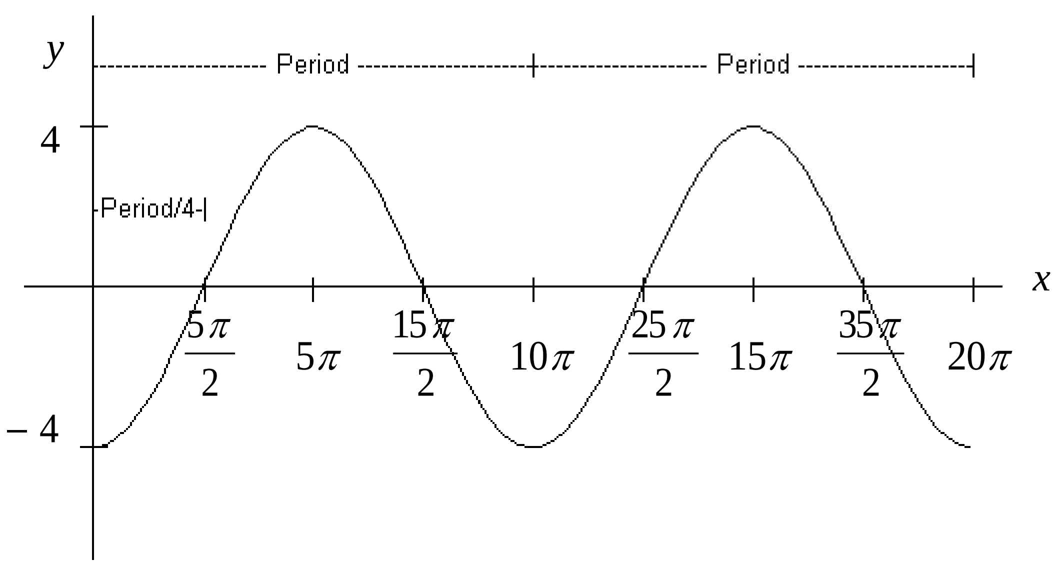
The graph of two cycles of $y = \sqrt{3}\cos 8x$ in blue compared with the graph of two cycles of $y = \cos x$ in red.

2. $y = -4\cos\left(\dfrac{x}{5}\right)$

Since the cosine function is being multiplied by a **negative** 4, then the graph will be inverted. Thus, we will need to draw two inverted cosine cycles.

Amplitude $= 4$ Period $= \dfrac{2\pi}{\frac{1}{5}} = 2\pi \cdot 5 = 10\pi$

$$\frac{1}{4} \cdot \text{period} = \frac{1}{4} \cdot 10\pi = \frac{10\pi}{4} = \frac{5\pi}{2}$$



Since the period is 10π, the first cycle ends at 10π and the second cycle ends at 20π. The other numbers on the *x*-axis were obtained by the following:

$$0 + \frac{5\pi}{2} = \frac{5\pi}{2}$$

$$\frac{5\pi}{2} + \frac{5\pi}{2} = \frac{10\pi}{2} = 5\pi$$

$$\frac{10\pi}{2} + \frac{5\pi}{2} = \frac{15\pi}{2}$$

Check: $\dfrac{15\pi}{2} + \dfrac{5\pi}{2} = \dfrac{20\pi}{2} = 10\pi$

$\dfrac{20\pi}{2} + \dfrac{5\pi}{2} = \dfrac{25\pi}{2}$ **OR** $\dfrac{5\pi}{2} + 10\pi = \dfrac{5\pi}{2} + \dfrac{20\pi}{2} = \dfrac{25\pi}{2}$

$\dfrac{25\pi}{2} + \dfrac{5\pi}{2} = \dfrac{30\pi}{2} = 15\pi$ **OR** $5\pi + 10\pi = 15\pi$

$\dfrac{30\pi}{2} + \dfrac{5\pi}{2} = \dfrac{35\pi}{2}$ **OR** $\dfrac{15\pi}{2} + 10\pi = \dfrac{15\pi}{2} + \dfrac{20\pi}{2} = \dfrac{35\pi}{2}$

Check: $\dfrac{35\pi}{2} + \dfrac{5\pi}{2} = \dfrac{40\pi}{2} = 20\pi$

The graph of two cycles of $y = -4\cos\left(\dfrac{x}{5}\right)$ in blue compared with the graph of two cycles of $y = -\cos x$ in red.

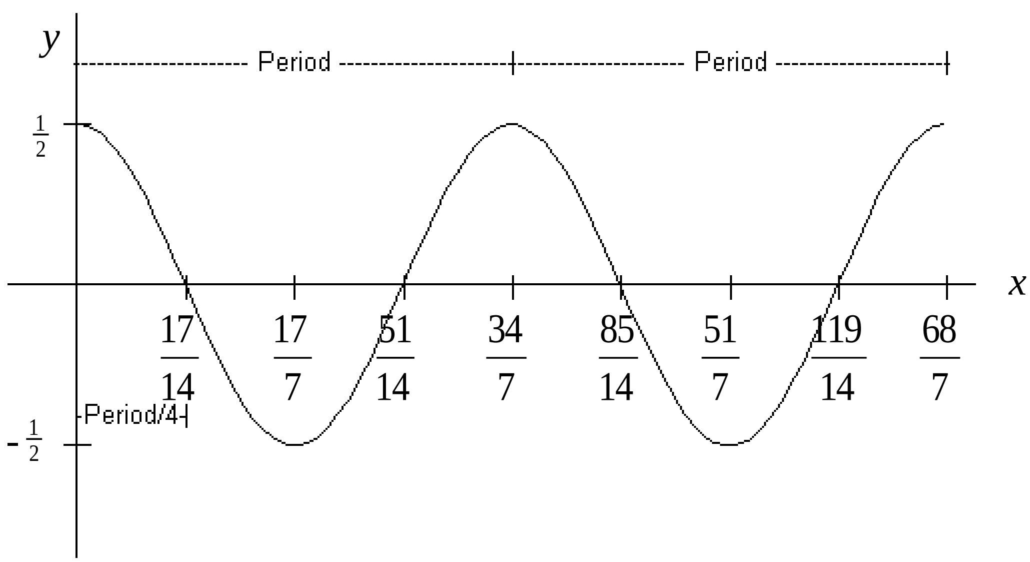
3. $y = \frac{1}{2}\cos\left(-\frac{7\pi x}{17}\right)$

NOTE: Since the cosine function is an even function, then $\cos\left(-\frac{7\pi x}{17}\right) = \cos\left(\frac{7\pi}{17}x\right)$. Thus, we have that

$$y = \frac{1}{2}\cos\left(-\frac{7\pi x}{17}\right) = \frac{1}{2}\cos\left(\frac{7\pi}{17}x\right)$$

Amplitude $= \frac{1}{2}$ Period $= \dfrac{2\pi}{\frac{7\pi}{17}} = 2\pi \cdot \frac{17}{7\pi} = 2 \cdot \frac{17}{7} = \frac{34}{7}$

$\frac{1}{4} \cdot$ period $= \frac{1}{4} \cdot \frac{34}{7} = \frac{1}{2} \cdot \frac{17}{7} = \frac{17}{14}$



Since the period is $\frac{34}{7}$, the first cycle ends at $\frac{34}{7}$ and the second cycle ends at $\frac{68}{7}$. The other numbers on the x-axis were obtained by the following:

$$0 + \frac{17}{14} = \frac{17}{14}$$

$$\frac{17}{14} + \frac{17}{14} = \frac{34}{14} = \frac{17}{7}$$

$$\frac{34}{14} + \frac{17}{14} = \frac{51}{14}$$

Check: $\frac{51}{14} + \frac{17}{14} = \frac{68}{14} = \frac{34}{7}$

$\frac{68}{14} + \frac{17}{14} = \frac{85}{14}$ **OR** $\frac{17}{14} + \frac{34}{7} = \frac{17}{14} + \frac{68}{14} = \frac{85}{14}$

$\frac{85}{14} + \frac{17}{14} = \frac{102}{14} = \frac{51}{7}$ **OR** $\frac{17}{7} + \frac{34}{7} = \frac{51}{7}$

$\frac{102}{14} + \frac{17}{14} = \frac{119}{14}$ **OR** $\frac{51}{14} + \frac{34}{7} = \frac{51}{14} + \frac{68}{14} = \frac{119}{14}$

Check: $\frac{119}{14} + \frac{17}{14} = \frac{136}{14} = \frac{68}{7}$

The graph of two cycles of $y = \frac{1}{2}\cos\left(-\frac{7\pi x}{17}\right)$ in blue compared with the graph of two cycles of $y = \cos x$ in red.

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3. SINE AND COSINE GRAPHS WITH PHASE SHIFTS

Definition A phase shift for a trigonometric function is a horizontal shift. That is, it is a shift with respect to the *x*-axis. Thus, the shift is either right or left.

NOTE: In order to identify a horizontal shift, hence, a phase shift, the coefficient of the *x* variable must be 1. If the coefficient is not 1, then you will need to factor out the coefficient.

Given the function $y = a \sin (bx + c)$, we may write this function as $y = a \sin \left[b \left(x + \frac{c}{b} \right) \right]$ by factoring out *b*.

The amplitude of this function is $|a|$ and the period is $\frac{2\pi}{|b|}$. The phase shift is $\frac{c}{b}$ units to the right if $\frac{c}{b} < 0$ or is $\frac{c}{b}$ units to the left if $\frac{c}{b} > 0$.

Similarly, given the function $y = a \cos (bx + c)$, we may write this function as $y = a \cos \left[b \left(x + \frac{c}{b} \right) \right]$ by factoring out *b*.

The amplitude of this function is $|a|$ and the period is $\frac{2\pi}{|b|}$. The phase shift is $\frac{c}{b}$ units to the right if $\frac{c}{b} < 0$ or is $\frac{c}{b}$ units to the left if $\frac{c}{b} > 0$.

Examples Sketch one cycle of the graph of the following functions.

1. $y = 3 \sin (2x - \pi)$

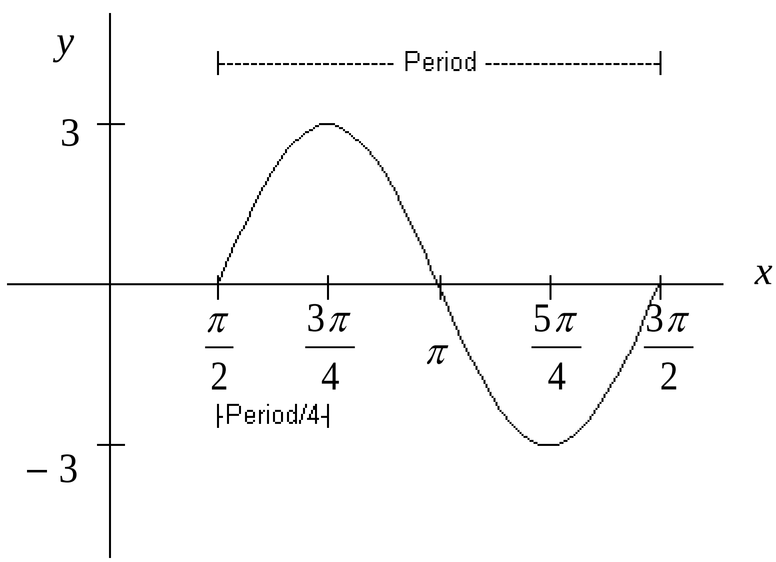
NOTE: Since the coefficient of the x variable is not 1, the phase shift is **not** π units to the right.

Since the coefficient of the x variable is 2, then we will need to factor the 2 out in order to identify the phase shift.

$$y = 3\sin(2x - \pi) = 3\sin\left[2\left(x - \frac{\pi}{2}\right)\right]$$

Amplitude = 3 Period $= \dfrac{2\pi}{2} = \pi$

Phase Shift: $\dfrac{\pi}{2}$ units to the right



Since the phase shift is $\dfrac{\pi}{2}$ units to the right, then the cycle starts at $\dfrac{\pi}{2}$. Since the period is π, then this cycle ends at $\dfrac{3\pi}{2}$ obtained by $\dfrac{\pi}{2} + \pi = \dfrac{\pi}{2} + \dfrac{2\pi}{2} = \dfrac{3\pi}{2}$.

The other numbers on the x-axis were obtained by the following:

$$\frac{1}{4} \cdot \text{period} = \frac{1}{4} \cdot \pi = \frac{\pi}{4}$$

The $\frac{3\pi}{4}$ was obtained by $\frac{\pi}{2} + \frac{\pi}{4} = \frac{2\pi}{4} + \frac{\pi}{4} = \frac{3\pi}{4}$. That is, we add $\frac{\pi}{4}$, which is one-fourth of the period, to the starting point of the cycle, which is $\frac{\pi}{2}$.

The π was obtained by $\frac{3\pi}{4} + \frac{\pi}{4} = \frac{4\pi}{4} = \pi$. That is, we add $\frac{\pi}{4}$, which is one-fourth of the period, to the next starting point of $\frac{3\pi}{4}$.

The $\frac{5\pi}{4}$ was obtained by $\frac{4\pi}{4} + \frac{\pi}{4} = \frac{5\pi}{4}$. That is, we add $\frac{\pi}{4}$, which is one-fourth of the period, to the next starting point of $\frac{4\pi}{4}$.

Check: $\frac{5\pi}{4} + \frac{\pi}{4} = \frac{6\pi}{4} = \frac{3\pi}{2}$

2. $y = 7 \sin\left(-\frac{x}{4} + \frac{2\pi}{9}\right)$

NOTE: Since the coefficient of the x variable is not 1, the phase shift is **not** $\frac{2\pi}{9}$ units to the left.

Since the coefficient of the x variable is $-\frac{1}{4}$, then we will need to factor the $-\frac{1}{4}$ out in order to identify the phase shift.

$$y = 7\sin\left(-\frac{x}{4} + \frac{2\pi}{9}\right) = 7\sin\left[-\frac{1}{4}\left(x - \frac{8\pi}{9}\right)\right]$$

NOTE: The $\frac{8\pi}{9}$ was obtained by $\frac{2\pi}{9} \div \frac{1}{4} = \frac{2\pi}{9} \cdot 4 = \frac{8\pi}{9}$.

Since the sine function is an odd function, then

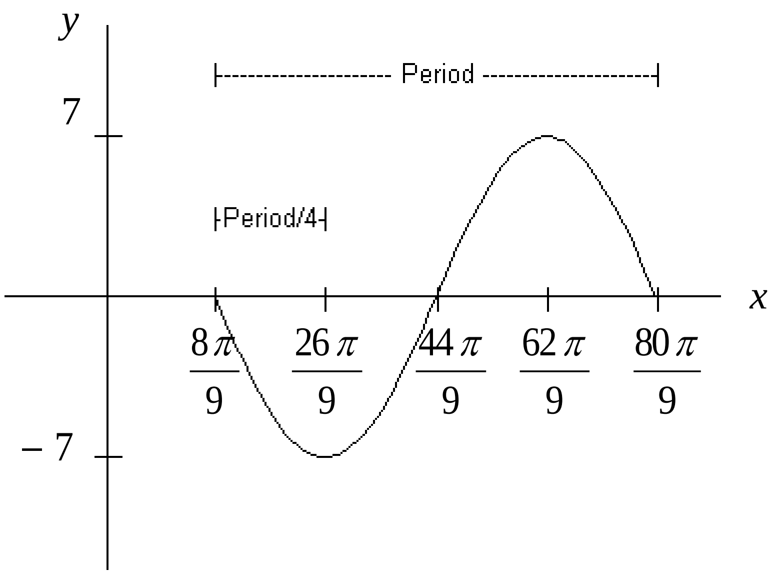
$$y = 7\sin\left[-\frac{1}{4}\left(x - \frac{8\pi}{9}\right)\right] = -7\sin\left[\frac{1}{4}\left(x - \frac{8\pi}{9}\right)\right]$$

Since the sine function is being multiplied by a **negative** 7, then the graph will be inverted. Thus, we will need to draw an inverted sine cycle.

Amplitude $= 7$

Period $= \dfrac{2\pi}{\frac{1}{4}} = 2\pi \cdot 4 = 8\pi$

Phase Shift: $\frac{8\pi}{9}$ units to the right



Since the phase shift is $\dfrac{8\pi}{9}$ units to the right, then the cycle starts at $\dfrac{8\pi}{9}$. Since the period is 8π, then this cycle ends at $\dfrac{80\pi}{9}$ obtained by $\dfrac{8\pi}{9} + 8\pi = \dfrac{8\pi}{9} + \dfrac{72\pi}{9} = \dfrac{80\pi}{9}$.

The other numbers on the *x*-axis were obtained by the following:

$$\frac{1}{4} \cdot \text{period} = \frac{1}{4} \cdot 8\pi = \frac{8\pi}{4} = 2\pi$$

The $\dfrac{26\pi}{9}$ was obtained by $\dfrac{8\pi}{9} + 2\pi = \dfrac{8\pi}{9} + \dfrac{18\pi}{9} = \dfrac{26\pi}{9}$. That is, we add 2π, which is one-fourth of the period, to the starting point of the cycle, which is $\dfrac{8\pi}{9}$.

The $\dfrac{44\pi}{9}$ was obtained by $\dfrac{26\pi}{9} + \dfrac{18\pi}{9} = \dfrac{44\pi}{9}$. That is, we add $2\pi = \dfrac{18\pi}{9}$, which is one-fourth of the period, to the next starting point of $\dfrac{26\pi}{9}$.

The $\dfrac{62\pi}{9}$ was obtained by $\dfrac{44\pi}{9} + \dfrac{18\pi}{9} = \dfrac{62\pi}{9}$. That is, we add $2\pi = \dfrac{18\pi}{9}$, which is one-fourth of the period, to the next starting point of $\dfrac{44\pi}{9}$.

Check: $\dfrac{62\pi}{9} + \dfrac{18\pi}{9} = \dfrac{80\pi}{9}$

3. $y = \sqrt{6}\sin\left(3x + \frac{\pi}{4}\right)$

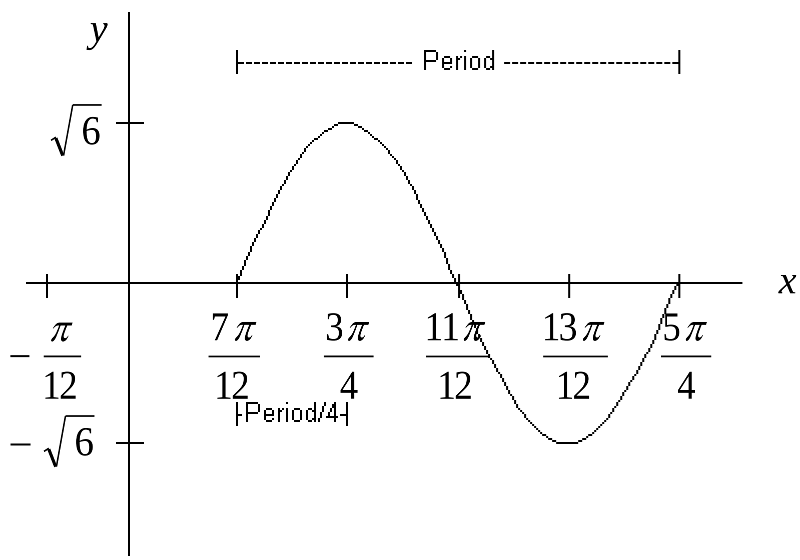
NOTE: Since the coefficient of the x variable is not 1, the phase shift is **not** $\frac{\pi}{4}$ units to the left.

Since the coefficient of the x variable is 3, then we will need to factor the 3 out in order to identify the phase shift.

$$y = \sqrt{6}\sin\left(3x + \frac{\pi}{4}\right) = \sqrt{6}\sin\left[3\left(x + \frac{\pi}{12}\right)\right]$$

Amplitude $= \sqrt{6}$ Period $= \frac{2\pi}{3}$

Phase Shift: $\frac{\pi}{12}$ units to the left



Since the phase shift is $\frac{\pi}{12}$ units to the left, then the cycle starts at $-\frac{\pi}{12}$. Since the period is $\frac{2\pi}{3}$, then this cycle ends at $\frac{7\pi}{12}$ obtained by

$-\dfrac{\pi}{12} + \dfrac{2\pi}{3} = -\dfrac{\pi}{12} + \dfrac{8\pi}{12} = \dfrac{7\pi}{12}$. This cycle starts to the left of the y-axis and finishes to the right of the y-axis. We are **sketching** a graph that goes up, down, and up again. If we allow our **sketch** to cross the y-axis, our picture will probably contain misinformation about where the actual graph crosses the y-axis. We do not want our picture to have misinformation in it. Our sketches have not been drawn to scale, but all the numbers on the x- and y-axes have been correct.

The sketch, that we draw, does not have to cross the y-axis. We know that where one cycle of the graph ends, another cycle begins.

In this problem, our first cycle ends at $\dfrac{7\pi}{12}$. Let's sketch the second cycle that begins at $\dfrac{7\pi}{12}$. This second cycle will end at $\dfrac{5\pi}{4}$ obtained by $\dfrac{7\pi}{12} + \dfrac{8\pi}{12} = \dfrac{15\pi}{12} = \dfrac{5\pi}{4}$. That is, will add the period of $\dfrac{2\pi}{3} = \dfrac{8\pi}{12}$ to the starting point of the second cycle, which is $\dfrac{7\pi}{12}$.

The other numbers on the x-axis were obtained by the following:

$$\frac{1}{4} \cdot \text{period} = \frac{1}{4} \cdot \frac{2\pi}{3} = \frac{1}{2} \cdot \frac{\pi}{3} = \frac{\pi}{6}$$

The $\dfrac{3\pi}{4}$ was obtained by $\dfrac{7\pi}{12} + \dfrac{\pi}{6} = \dfrac{7\pi}{12} + \dfrac{2\pi}{12} = \dfrac{9\pi}{12} = \dfrac{3\pi}{4}$. That is, we add $\dfrac{\pi}{6}$, which is one-fourth of the period, to the starting point of the second cycle, which is $\dfrac{7\pi}{12}$.

The $\dfrac{11\pi}{12}$ was obtained by $\dfrac{9\pi}{12} + \dfrac{2\pi}{12} = \dfrac{11\pi}{12}$. That is, we add $\dfrac{\pi}{6} = \dfrac{2\pi}{12}$, which is one-fourth of the period, to the next starting point of $\dfrac{9\pi}{12}$.

The $\dfrac{13\pi}{12}$ was obtained by $\dfrac{11\pi}{12} + \dfrac{2\pi}{12} = \dfrac{13\pi}{12}$. That is, we add $2\pi = \dfrac{18\pi}{9}$, which is one-fourth of the period, to the next starting point of $\dfrac{11\pi}{12}$.

Check: $\dfrac{13\pi}{12} + \dfrac{2\pi}{12} = \dfrac{15\pi}{12} = \dfrac{5\pi}{4}$

4. $y = -\dfrac{11}{8}\sin\left(-\dfrac{6x}{5} - \dfrac{12\pi}{7}\right)$

Since the coefficient of the x variable is $-\dfrac{6}{5}$, then we will need to factor the $-\dfrac{6}{5}$ out in order to identify the phase shift.

$$y = -\frac{11}{8}\sin\left(-\frac{6x}{5} - \frac{12\pi}{7}\right) = -\frac{11}{8}\sin\left[-\frac{6}{5}\left(x + \frac{10\pi}{7}\right)\right]$$

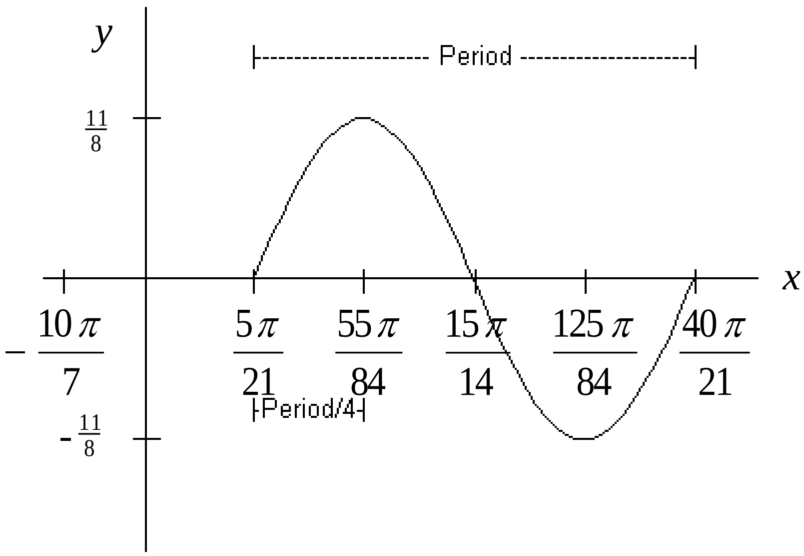
NOTE: The $\dfrac{10\pi}{7}$ was obtained by $\dfrac{12\pi}{7} \div \dfrac{6}{5} = \dfrac{12\pi}{7} \cdot \dfrac{5}{6} = \dfrac{2\pi}{7} \cdot \dfrac{5}{1} = \dfrac{10\pi}{7}$.

Since the sine function is an odd function, then

$$y = -\frac{11}{8}\sin\left[-\frac{6}{5}\left(x + \frac{10\pi}{7}\right)\right] = \frac{11}{8}\sin\left[\frac{6}{5}\left(x + \frac{10\pi}{7}\right)\right]$$

Amplitude $= \dfrac{11}{8}$ Period $= \dfrac{2\pi}{\frac{6}{5}} = 2\pi \cdot \dfrac{5}{6} = \pi \cdot \dfrac{5}{3} = \dfrac{5\pi}{3}$

Phase Shift: $\dfrac{10\pi}{7}$ units to the left



Since the phase shift is $\dfrac{10\pi}{7}$ units to the left, then the cycle starts at $-\dfrac{10\pi}{7}$. Since the period is $\dfrac{5\pi}{3}$, then this cycle ends at $\dfrac{5\pi}{21}$ obtained by $-\dfrac{10\pi}{7} + \dfrac{5\pi}{3} = -\dfrac{30\pi}{21} + \dfrac{35\pi}{21} = \dfrac{5\pi}{21}$. This cycle starts to the left of the *y*-axis and finishes to the right of the *y*-axis. Again, since we are sketching the graph of this function, we do not want our sketch to cross the *y*-axis because our picture will probably contain misinformation about where the actual graph crosses the *y*-axis.

In this problem, our first cycle ends at $\dfrac{5\pi}{21}$. So, let's sketch the second cycle that begins at $\dfrac{5\pi}{21}$. This second cycle will end at $\dfrac{40\pi}{21}$ obtained by

$\frac{5\pi}{21} + \frac{35\pi}{21} = \frac{40\pi}{21}$. That is, will add the period of $\frac{5\pi}{3} = \frac{35\pi}{21}$ to the starting point of the second cycle, which is $\frac{5\pi}{21}$.

The other numbers on the *x*-axis were obtained by the following:

$\frac{1}{4} \cdot \text{period} = \frac{1}{4} \cdot \frac{5\pi}{3} = \frac{5\pi}{12}$

The $\frac{55\pi}{84}$ was obtained by $\frac{5\pi}{21} + \frac{5\pi}{12} = \frac{20\pi}{84} + \frac{35\pi}{84} = \frac{55\pi}{84}$. That is, we add $\frac{5\pi}{12}$, which is one-fourth of the period, to the starting point of the second cycle, which is $\frac{5\pi}{21}$.

The $\frac{15\pi}{14}$ was obtained by $\frac{55\pi}{84} + \frac{35\pi}{84} = \frac{90\pi}{84} = \frac{45\pi}{42} = \frac{15\pi}{14}$. That is, we add $\frac{5\pi}{12} = \frac{35\pi}{84}$, which is one-fourth of the period, to the next starting point of $\frac{55\pi}{84}$.

The $\frac{125\pi}{84}$ was obtained by $\frac{90\pi}{84} + \frac{35\pi}{84} = \frac{125\pi}{84}$. That is, we add $\frac{5\pi}{12} = \frac{35\pi}{84}$, which is one-fourth of the period, to the next starting point of $\frac{90\pi}{84}$.

Check: $\frac{125\pi}{84} + \frac{35\pi}{84} = \frac{160\pi}{84} = \frac{40\pi}{21}$

5. $y = -2\cos\left(4x - \frac{\pi}{6}\right)$

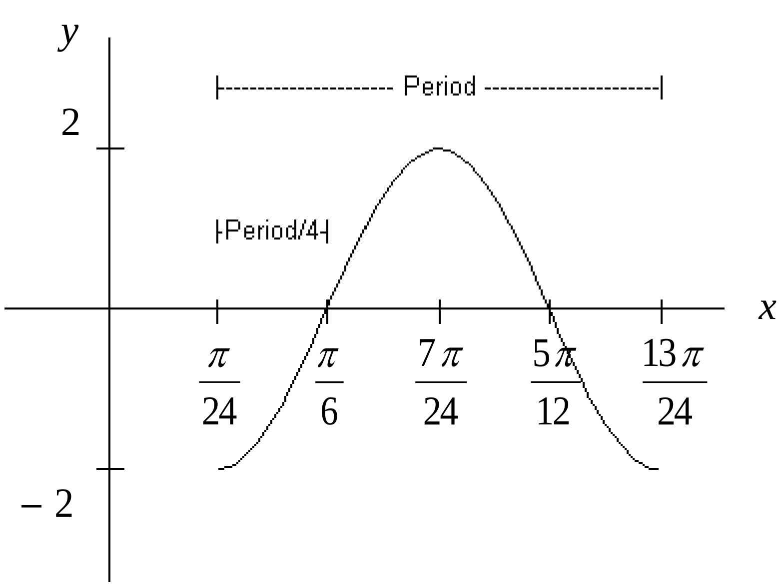
Since the coefficient of the *x* variable is 4, then we will need to factor the 4 out in order to identify the phase shift.

$$y = -2\cos\left(4x - \frac{\pi}{6}\right) = -2\cos\left[4\left(x - \frac{\pi}{24}\right)\right]$$

Since the cosine function is being multiplied by a **negative** 2, then the graph will be inverted. Thus, we will need to draw an inverted cosine cycle.

Amplitude $= 2$ Period $= \frac{2\pi}{4} = \frac{\pi}{2}$

Phase Shift: $\frac{\pi}{24}$ units to the right



Since the phase shift is $\dfrac{\pi}{24}$ units to the right, then the cycle starts at $\dfrac{\pi}{24}$. Since the period is $\dfrac{\pi}{2}$, then this cycle ends at $\dfrac{13\pi}{24}$ obtained by $\dfrac{\pi}{24} + \dfrac{\pi}{2} = \dfrac{\pi}{24} + \dfrac{12\pi}{24} = \dfrac{13\pi}{24}$.

The other numbers on the *x*-axis were obtained by the following:

$$\frac{1}{4} \cdot \text{period} = \frac{1}{4} \cdot \frac{\pi}{2} = \frac{\pi}{8}$$

The $\dfrac{\pi}{6}$ was obtained by $\dfrac{\pi}{24} + \dfrac{\pi}{8} = \dfrac{\pi}{24} + \dfrac{3\pi}{24} = \dfrac{4\pi}{24} = \dfrac{\pi}{6}$. That is, we add $\dfrac{\pi}{8}$, which is one-fourth of the period, to the starting point of the cycle, which is $\dfrac{\pi}{24}$.

The $\dfrac{7\pi}{24}$ was obtained by $\dfrac{4\pi}{24} + \dfrac{3\pi}{24} = \dfrac{7\pi}{24}$. That is, we add $\dfrac{\pi}{8} = \dfrac{3\pi}{24}$, which is one-fourth of the period, to the next starting point of $\dfrac{4\pi}{24}$.

The $\dfrac{5\pi}{6}$ was obtained by $\dfrac{7\pi}{24} + \dfrac{3\pi}{24} = \dfrac{10\pi}{24} = \dfrac{5\pi}{12}$. That is, we add $\dfrac{\pi}{8} = \dfrac{3\pi}{24}$, which is one-fourth of the period, to the next starting point of $\dfrac{7\pi}{24}$.

Check: $\dfrac{10\pi}{24} + \dfrac{3\pi}{24} = \dfrac{13\pi}{24}$

6. $\quad y = \dfrac{1}{6}\cos\left(\dfrac{\pi x}{3} + \dfrac{5\pi}{8}\right)$

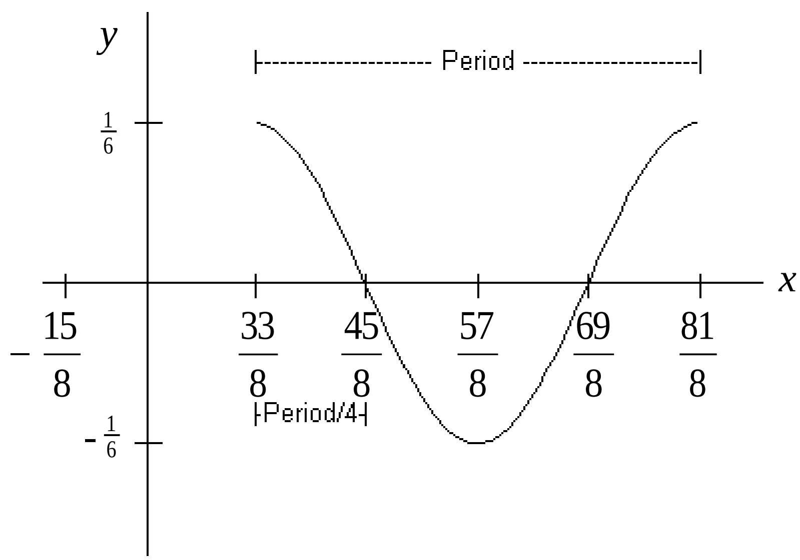
Since the coefficient of the x variable is $\frac{\pi}{3}$, then we will need to factor the $\frac{\pi}{3}$ out in order to identify the phase shift.

$$y = \frac{1}{6}\cos\left(\frac{\pi x}{3} + \frac{5\pi}{8}\right) = \frac{1}{6}\cos\left[\frac{\pi}{3}\left(x + \frac{15}{8}\right)\right]$$

NOTE: The $\frac{15}{8}$ was obtained by $\frac{5\pi}{8} \div \frac{\pi}{3} = \frac{5\pi}{8} \cdot \frac{3}{\pi} = \frac{5}{8} \cdot \frac{3}{1} = \frac{15}{8}$.

Amplitude $= \frac{1}{6}$ Period $= \frac{2\pi}{\frac{\pi}{3}} = 2\pi \cdot \frac{3}{\pi} = 2 \cdot \frac{3}{1} = 6$

Phase Shift: $\frac{15}{8}$ units to the left



Since the phase shift is $\frac{15}{8}$ units to the left, then the cycle starts at $-\frac{15}{8}$. Since the period is 6, then this cycle ends at $\frac{33}{8}$ obtained by $-\frac{15}{8} + 6 = -\frac{15}{8} + \frac{48}{8} = \frac{33}{8}$. This cycle starts to the left of the y-axis and finishes to the right of the *y*-axis. Again, since we are sketching the graph of this function, we do not want our sketch to cross the *y*-axis because our picture will probably contain misinformation about where the actual graph crosses the *y*-axis.

In this problem, our first cycle ends at $\frac{33}{8}$. So, let's sketch the second cycle that begins at $\frac{33}{8}$. This second cycle will end at $\frac{81}{8}$ obtained by $\frac{33}{8} + \frac{48}{8} = \frac{81}{8}$. That is, will add the period of $6 = \frac{48}{8}$ to the starting point of the second cycle, which is $\frac{33}{8}$.

The other numbers on the *x*-axis were obtained by the following:

$$\frac{1}{4} \cdot \text{period} = \frac{1}{4} \cdot 6 = \frac{6}{4} = \frac{3}{2}$$

The $\frac{45}{8}$ was obtained by $\frac{33}{8} + \frac{3}{2} = \frac{33}{8} + \frac{12}{8} = \frac{45}{8}$. That is, we add $\frac{5\pi}{12}$, which is one-fourth of the period, to the starting point of the second cycle, which is $\frac{33}{8}$.

The $\frac{57}{8}$ was obtained by $\frac{45}{8} + \frac{12}{8} = \frac{57}{8}$. That is, we add $\frac{3}{2} = \frac{12}{8}$, which is one-fourth of the period, to the next starting point of $\frac{45}{8}$.

The $\frac{69}{8}$ was obtained by $\frac{57}{8} + \frac{12}{8} = \frac{69}{8}$. That is, we add $\frac{3}{2} = \frac{12}{8}$, which is one-fourth of the period, to the next starting point of $\frac{57}{8}$.

Check: $\dfrac{69}{8} + \dfrac{12}{8} = \dfrac{81}{8}$

7. $y = \cos\left(\dfrac{2\pi}{5} - x\right)$

Since the coefficient of the x variable is -1, then we will need to factor the -1 out in order to identify the phase shift.

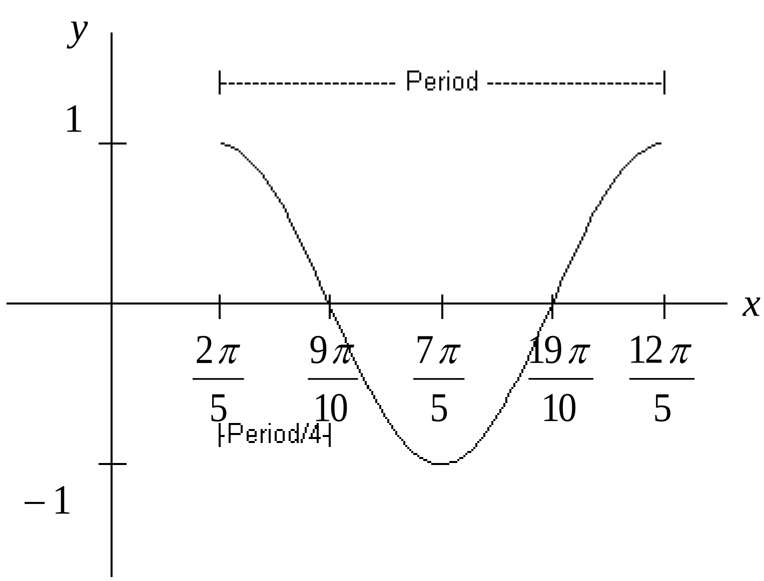
$$y = \cos\left(\frac{2\pi}{5} - x\right) = \cos\left(-x + \frac{2\pi}{5}\right) = \cos\left[-\left(x - \frac{2\pi}{5}\right)\right]$$

Since the cosine function is an even function, then

$$y = \cos\left[-\left(x - \frac{2\pi}{5}\right)\right] = \cos\left(x - \frac{2\pi}{5}\right)$$

Amplitude $= 1$ Period $= 2\pi$

Phase Shift: $\dfrac{2\pi}{5}$ units to the right



Since the phase shift is $\dfrac{2\pi}{5}$ units to the right, then the cycle starts at $\dfrac{2\pi}{5}$. Since the period is 2π, then this cycle ends at $\dfrac{12\pi}{5}$ obtained by $\dfrac{2\pi}{5} + 2\pi = \dfrac{2\pi}{5} + \dfrac{10\pi}{5} = \dfrac{12\pi}{5}$.

The other numbers on the *x*-axis were obtained by the following:

$$\frac{1}{4} \cdot \text{period} = \frac{2\pi}{4} = \frac{\pi}{2}$$

The $\dfrac{9\pi}{10}$ was obtained by $\dfrac{2\pi}{5} + \dfrac{\pi}{2} = \dfrac{4\pi}{10} + \dfrac{5\pi}{10} = \dfrac{9\pi}{10}$. That is, we add $\dfrac{\pi}{2}$, which is one-fourth of the period, to the starting point of the cycle, which is $\dfrac{2\pi}{5}$.

The $\dfrac{7\pi}{5}$ was obtained by $\dfrac{9\pi}{10} + \dfrac{5\pi}{10} = \dfrac{14\pi}{10} = \dfrac{7\pi}{5}$. That is, we add $\dfrac{\pi}{2} = \dfrac{5\pi}{10}$, which is one-fourth of the period, to the next starting point of $\dfrac{9\pi}{10}$.

The $\dfrac{19\pi}{10}$ was obtained by $\dfrac{14\pi}{10} + \dfrac{5\pi}{10} = \dfrac{19\pi}{10}$. That is, we add $\dfrac{\pi}{2} = \dfrac{5\pi}{10}$, which is one-fourth of the period, to the next starting point of $\dfrac{14\pi}{10}$.

Check: $\dfrac{19\pi}{10} + \dfrac{5\pi}{10} = \dfrac{24\pi}{10} = \dfrac{12\pi}{5}$

Back to Topics List

4. SECANT AND COSECANT GRAPHS

Given the function $y = a\csc(bx + c)$, we may write this function as $y = a\csc\left[b\left(x + \frac{c}{b}\right)\right]$ by factoring out b. The cosecant function does not have an amplitude, the period is $\frac{2\pi}{|b|}$, and the phase shift is $\frac{c}{b}$ units to the right if $\frac{c}{b} < 0$ or is $\frac{c}{b}$ units to the left if $\frac{c}{b} > 0$. In order to obtain a sketch of the graph of the cosecant function, you will make use of the sketch of the graph of sine function. First sketch the graph of $y = a\sin(bx + c)$ and then locate the *x*-intercepts of the sketch. These are the locations of the vertical asymptotes of the cosecant function. Draw these vertical asymptotes and then use the sketch of the graph of the sine function to sketch the graph of the cosecant function.

Similarly, given the function $y = a\sec(bx + c)$, we may write this function as $y = a\sec\left[b\left(x + \frac{c}{b}\right)\right]$ by factoring out b. The secant function does not have an amplitude, the period is $\frac{2\pi}{|b|}$, and the phase shift is $\frac{c}{b}$ units to the right if $\frac{c}{b} < 0$ or is $\frac{c}{b}$ units to the left if $\frac{c}{b} > 0$. In order to obtain a sketch of the graph of the secant function, you will make use of the sketch of the graph of cosine function. First sketch the graph of $y = a\cos(bx + c)$ and then locate the *x*-intercepts of the sketch. These are the locations of the vertical asymptotes of the secant function. Draw these vertical asymptotes and then use the sketch of the graph of the cosine function to sketch the graph of the secant function.

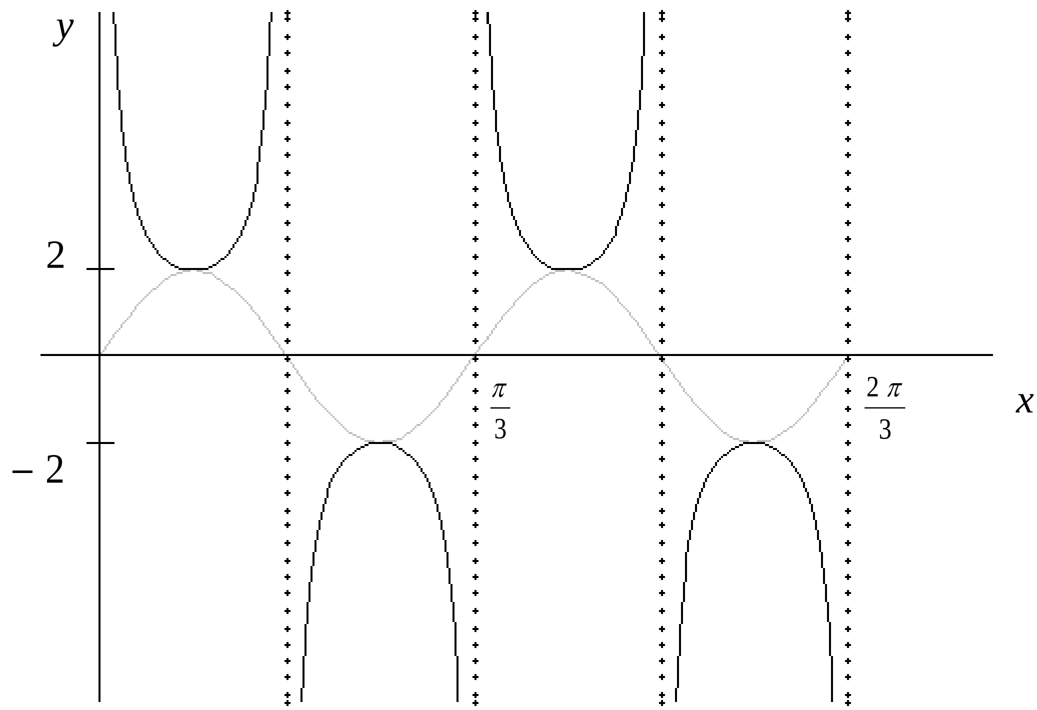
Examples Sketch two cycles of the graph of the following functions. Label the numbers on the *y*-axis. On the *x*-axis, only label where the cycles begin and end.

1. $y = 2\csc 6x$

   First, sketch the graph of $y = 2\sin 6x$. For this sine function, we have the following:

Amplitude = 2 Period = $\frac{2\pi}{6} = \frac{\pi}{3}$ Phase Shift: None

Since there is no phase shift and the period is $\frac{\pi}{3}$, then the first cycle starts at 0 and end at $\frac{\pi}{3}$. The second cycle starts at $\frac{\pi}{3}$ and ends at $\frac{2\pi}{3}$.



2. $y = \frac{8}{5}\csc\left(-\frac{\pi x}{4}\right)$

First, sketch the graph of $y = \frac{8}{5}\sin\left(-\frac{\pi x}{4}\right)$. Since the sine function is an odd function, then $\sin\left(-\frac{\pi x}{4}\right) = -\sin\left(\frac{\pi}{4}x\right)$. Thus, we have that

$$y = \frac{8}{5}\sin\left(-\frac{\pi x}{4}\right) = -\frac{8}{5}\sin\left(\frac{\pi}{4}x\right)$$

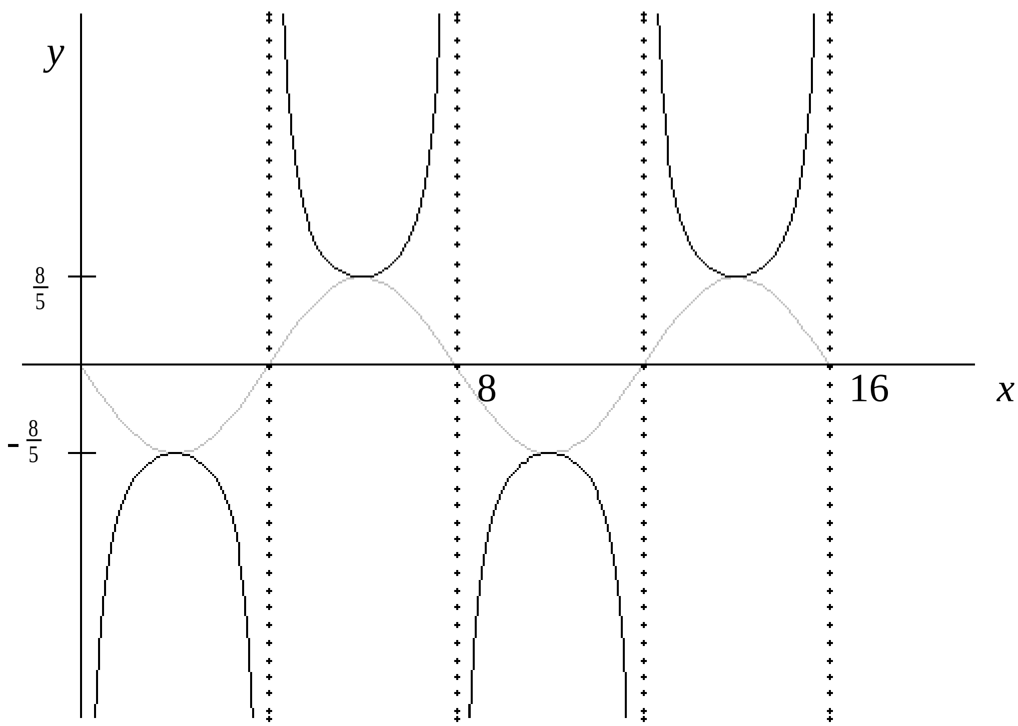
Since the sine function is being multiplied by a **negative** $\frac{8}{5}$, then the graph will be inverted. Thus, we will need to draw two inverted sine cycles.

For this sine function, we have the following:

Amplitude $= \frac{8}{5}$ Period $= \dfrac{2\pi}{\frac{\pi}{4}} = 2\pi \cdot \frac{4}{\pi} = 2 \cdot \frac{4}{1} = 8$

Phase Shift: None

Since there is no phase shift and the period is 8, then the first cycle starts at 0 and end at 8. The second cycle starts at 8 and ends at 16.



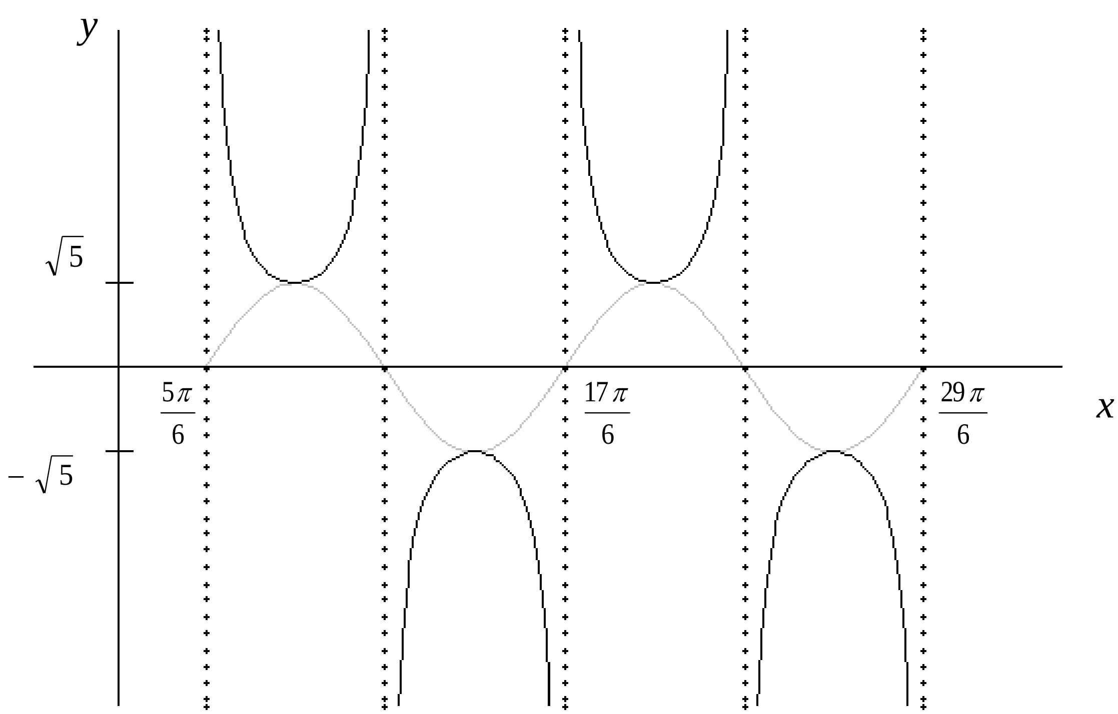
3. $y = \sqrt{5} \csc\left(x - \frac{5\pi}{6} \right)$

First, sketch the graph of $y = \sqrt{5}\sin\left(x - \frac{5\pi}{6}\right)$. Notice the coefficient of the x variable is 1. For this sine function, we have the following:

Amplitude $= \sqrt{5}$ Period $= 2\pi$

Phase Shift: $\frac{5\pi}{6}$ units to the right

Since the phase shift is $\frac{5\pi}{6}$ units to the right, then the first cycle starts at $\frac{5\pi}{6}$. Since the period is 2π, then this first cycle ends at $\frac{17\pi}{6}$, obtained by $\frac{5\pi}{6} + 2\pi = \frac{5\pi}{6} + \frac{12\pi}{6} = \frac{17\pi}{6}$, and the second cycle ends at $\frac{29\pi}{6}$, obtained by $\frac{17\pi}{6} + \frac{12\pi}{6} = \frac{29\pi}{6}$.



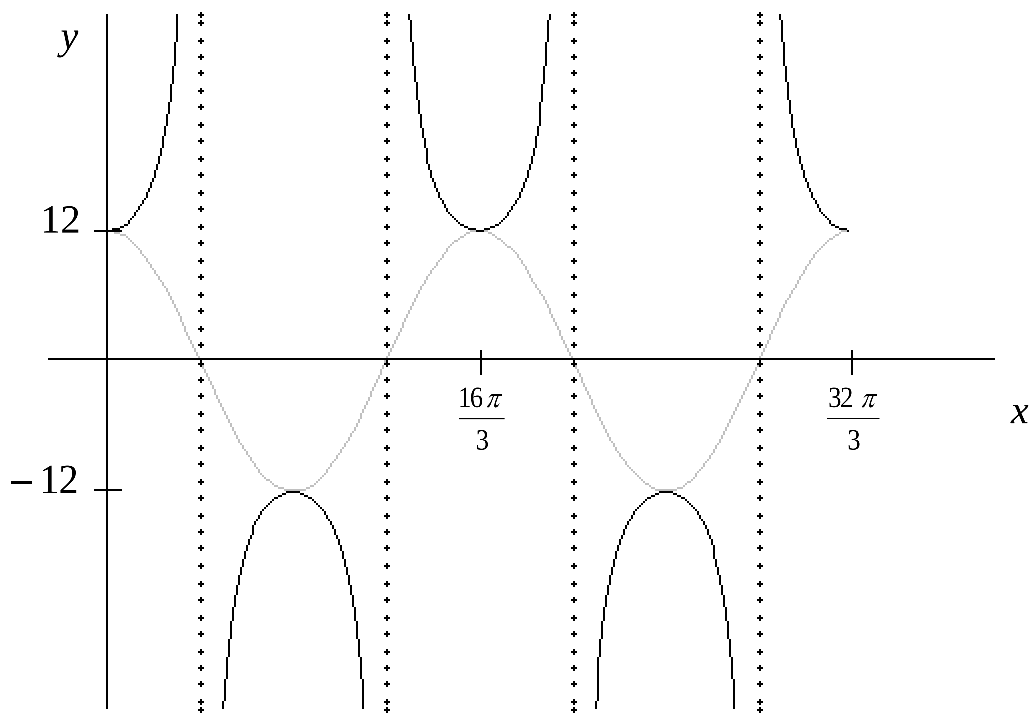
4. $y = 12 \sec\left(\frac{3x}{8}\right)$

First, sketch the graph of $y = 12 \cos\left(\frac{3}{8}x\right)$. For this cosine function, we have the following:

Amplitude $= 12$ Period $= \dfrac{2\pi}{\frac{3}{8}} = 2\pi \cdot \dfrac{8}{3} = \dfrac{16\pi}{3}$

Phase Shift: None

Since there is no phase shift and the period is $\frac{16\pi}{3}$, then the first cycle starts at 0 and end at $\frac{16\pi}{3}$. The second cycle starts at $\frac{16\pi}{3}$ and ends at $\frac{32\pi}{3}$.



5. $y = -\sec \pi x$

First, sketch the graph of $y = -\cos \pi x$.

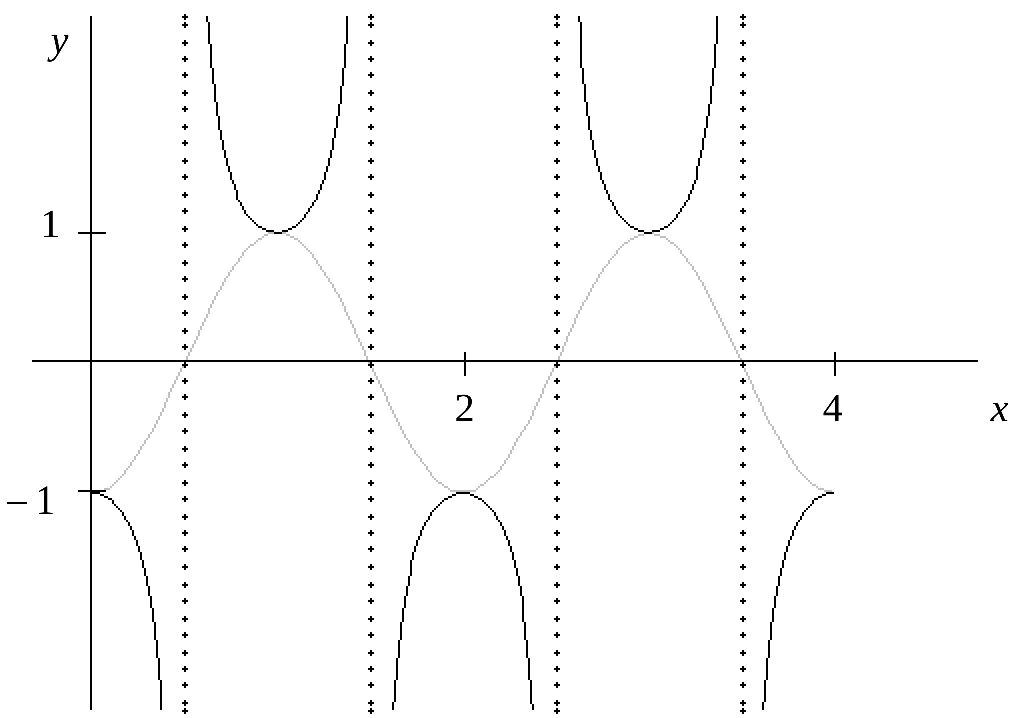
Since the cosine function is being multiplied by a **negative** 1, then the graph will be inverted. Thus, we will need to draw two inverted cosine cycles.

For this cosine function, we have the following:

Amplitude $= 1$ Period $= \dfrac{2\pi}{\pi} = 2$

Phase Shift: None

Since there is no phase shift and the period is 2, then the first cycle starts at 0 and end at 2. The second cycle starts at 2 and ends at 4.



6. $y = 5 \sec\left(7x + \dfrac{3\pi}{4}\right)$

First, sketch the graph of $y = 5 \cos\left(7x + \dfrac{3\pi}{4}\right)$. Since the coefficient of the x variable is 7, then we will need to factor the 7 out in order to identify the phase shift. Thus, we have that

$$y = 5\cos\left(7x + \frac{3\pi}{4}\right) = 5\cos\left[7\left(x + \frac{3\pi}{28}\right)\right]$$

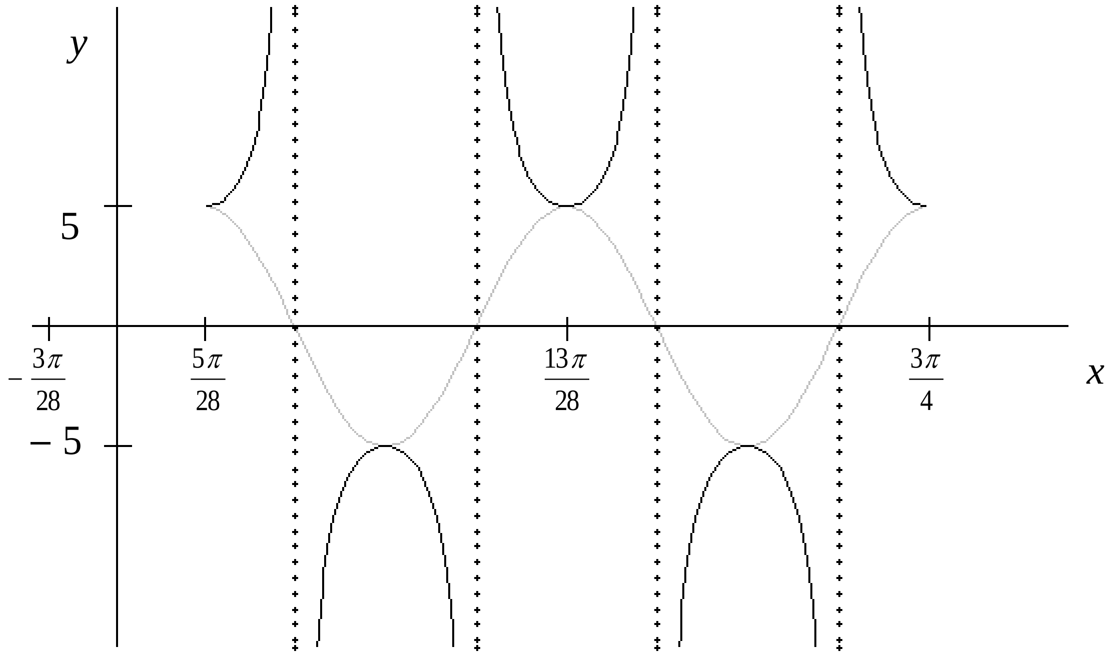
For this cosine function, we have the following:

Amplitude = 5 Period = $\frac{2\pi}{7}$

Phase Shift: $\frac{3\pi}{28}$ units to the left

Since the phase shift is $\frac{3\pi}{28}$ units to the left, then the first cycle starts at $-\frac{3\pi}{28}$. Since the period is $\frac{2\pi}{7}$, then this first cycle ends at $\frac{5\pi}{28}$, obtained by $-\frac{3\pi}{28} + \frac{2\pi}{7} = -\frac{3\pi}{28} + \frac{8\pi}{28} = \frac{5\pi}{28}$. This cycle starts to the left of the *y*-axis and finishes to the right of the *y*-axis. Since we are sketching the graph of this function, we do not want our sketch to cross the *y*-axis because our picture will probably contain misinformation about where the actual graph crosses the *y*-axis.

In this problem, our first cycle ends at $\frac{5\pi}{28}$. So, let's sketch the second cycle that begins at $\frac{5\pi}{28}$ as our first cycle. This cycle will end at $\frac{13\pi}{28}$ obtained by $\frac{5\pi}{28} + \frac{8\pi}{28} = \frac{13\pi}{28}$. The next cycle will end at $\frac{3\pi}{4}$, obtained by $\frac{13\pi}{28} + \frac{8\pi}{28} = \frac{21\pi}{28} = \frac{3\pi}{4}$.



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5. TANGENT GRAPHS

Example Find the x-intercepts of the graph of $y = \tan x$ in the interval $[-2\pi, 2\pi]$.

NOTE: The interval $[-2\pi, 2\pi]$ of angles are the angles going one time around the Unit Circle clockwise for the subinterval $[-2\pi, 0]$ and the angles going one time around the Unit Circle counterclockwise for the subinterval $[0, 2\pi]$.

To find the x-intercepts, set y equal to 0: $\tan x = 0$.

Since $\tan x = \dfrac{\sin x}{\cos x}$, then $\tan x = 0 \Rightarrow \dfrac{\sin x}{\cos x} = 0$.

Since a fraction can only equal zero when the numerator of the fraction equals zero, then $\dfrac{\sin x}{\cos x} = 0 \Rightarrow \sin x = 0$. By Unit Circle Trigonometry, we are looking for angles in the interval $[-2\pi, 2\pi]$ that intersect the Unit Circle so that the y-

coordinate of the point of intersection is 0. Going around the Unit Circle clockwise, these angles are
$x = 0, -\pi, -2\pi$. Going around the Unit Circle counterclockwise, these angles are $x = 0, \pi, 2\pi$.

Thus, the *x*-intercepts of the graph of $y = \tan x$ in the interval $[-2\pi, 2\pi]$ are the points $(-2\pi, 0)$, $(-\pi, 0)$, $(0, 0)$, $(\pi, 0)$, and $(2\pi, 0)$.

Example Find the vertical asymptotes of the graph of $y = \tan x$ in the interval $[-2\pi, 2\pi]$.

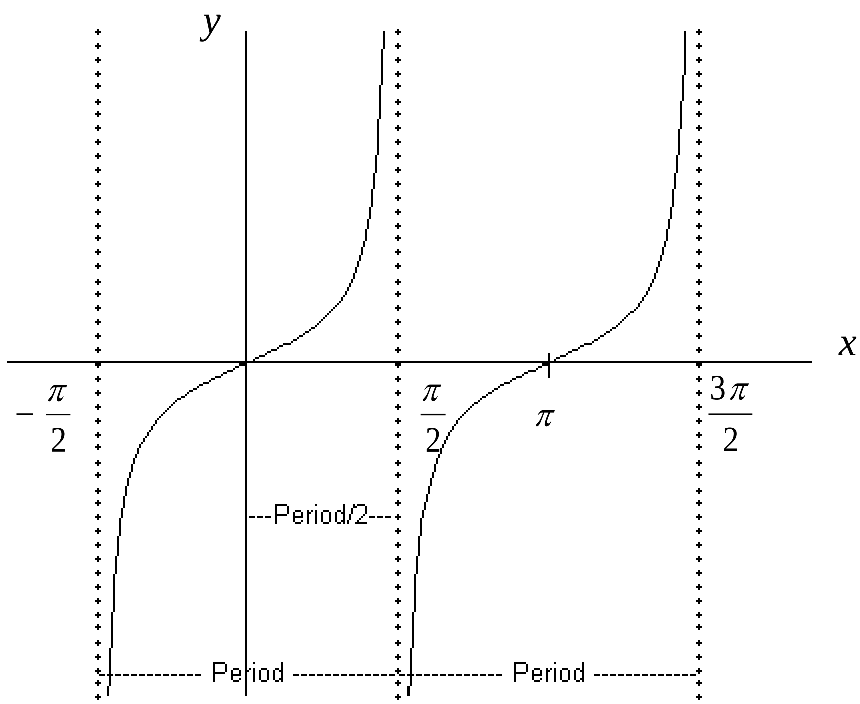
NOTE: The interval $[-2\pi, 2\pi]$ of angles are the angles going one time around the Unit Circle clockwise for the subinterval $[-2\pi, 0]$ and the angles going one time around the Unit Circle counterclockwise for the subinterval $[0, 2\pi]$.

Since $\tan x = \dfrac{\sin x}{\cos x}$, then the vertical asymptotes will occur where the denominator of this fraction is equal to zero. Thus, we want to solve the equation $\cos x = 0$ in the interval $[-2\pi, 2\pi]$.

By Unit Circle Trigonometry, we are looking for angles in the interval $[-2\pi, 2\pi]$ that intersect the Unit Circle so that the *x*-coordinate of the point of intersection is 0. Going around the Unit Circle clockwise, these angles are $x = -\dfrac{\pi}{2}, -\dfrac{3\pi}{2}$. Going around the Unit Circle counterclockwise, these angles are $x = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$.

Thus, the vertical asymptotes of the graph of $y = \tan x$ in the interval $[-2\pi, 2\pi]$ are $x = -\dfrac{3\pi}{2}$, $x = -\dfrac{\pi}{2}$, $x = \dfrac{\pi}{2}$, and $x = \dfrac{3\pi}{2}$.

Example Sketch two cycles of the graph of $y = \tan x$ using the *x*-intercepts and vertical asymptotes of the function.



The function $y = \tan x$ does not have an amplitude and the period of the function is π.

In general, given the function $y = a \tan (bx + c)$, we may write this function as $y = a \tan \left[b\left(x + \frac{c}{b} \right) \right]$ by factoring out *b*. The tangent function does not have an amplitude, the period is $\frac{\pi}{|b|}$, and the phase shift is $\frac{c}{b}$ units to the right if $\frac{c}{b} < 0$ or is $\frac{c}{b}$ units to the left if $\frac{c}{b} > 0$.

<u>Theorem</u> The tangent function is an odd function. That is, $\tan(-\theta) = -\tan \theta$ for all θ in the domain of the function.

A sketch of the graph of the tangent function can be obtained from the *x*-intercepts and vertical asymptotes of the function. Two consecutive vertical asymptotes are a distance of the **period** from each other. The *x*-coordinates of the *x*-intercepts are midway between two consecutive vertical asymptotes. That is, the x-coordinates of the x-intercepts are the midpoint of two consecutive vertical asymptotes.

So to sketch the graph of a tangent function, you only need to know where the first vertical asymptote is located. Then use the period to find the next consecutive vertical asymptote. For an unshifted tangent function, the first two consecutive vertical asymptotes are symmetric about the y-axis. For a shifted tangent function, first shift one vertical asymptote for the unshifted graph. Once you have the vertical asymptotes, you find the x-coordinates of the x-intercepts by finding the midpoint of two consecutive vertical asymptotes.

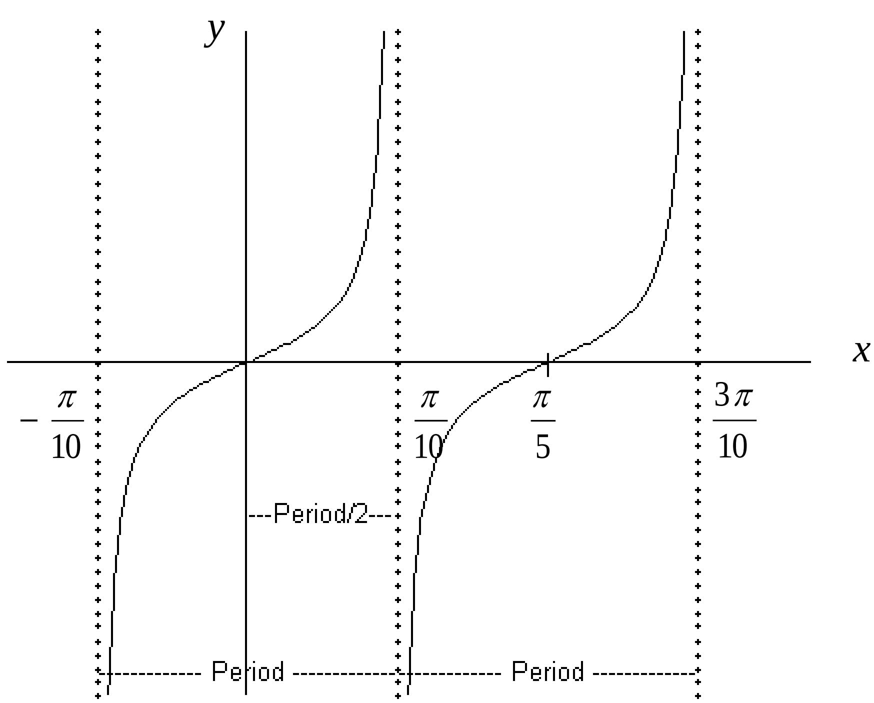
Recall the midpoint of the numbers a and b is the number $\dfrac{a + b}{2}$.

Examples Sketch two cycles of the graph of the following functions.

1. $y = 8 \tan 5x$

Amplitude: None   Period $= \dfrac{\pi}{5}$ Phase Shift: None

$\dfrac{1}{2}$ period $= \dfrac{\pi}{10}$



The $\dfrac{3\pi}{10}$ was obtained by $\dfrac{\pi}{10} + \dfrac{\pi}{5} = \dfrac{\pi}{10} + \dfrac{2\pi}{10} = \dfrac{3\pi}{10}$. That is, we add $\dfrac{\pi}{5}$, which is the period, to the starting point of $\dfrac{\pi}{10}$, which is where the first vertical asymptote to the right of the *y*-axis crosses the *x*-axis.

The $\dfrac{\pi}{5}$ was obtained by $\dfrac{1}{2}\left(\dfrac{\pi}{10} + \dfrac{3\pi}{10}\right) = \dfrac{1}{2}\left(\dfrac{4\pi}{10}\right) = \dfrac{1}{2}\left(\dfrac{2\pi}{5}\right) = \dfrac{\pi}{5}$. That is, we found the midpoint of $\dfrac{\pi}{10}$ and $\dfrac{3\pi}{10}$. Of course, you could also use the fact that two consecutive *x*-intercepts are a distance of the period from each other. Since the first *x*-intercept is 0 and the period is $\dfrac{\pi}{5}$, then the second *x*-intercept is $\dfrac{\pi}{5}$.

2. $y = -\sqrt{21}\tan\left(\dfrac{9x}{8}\right)$

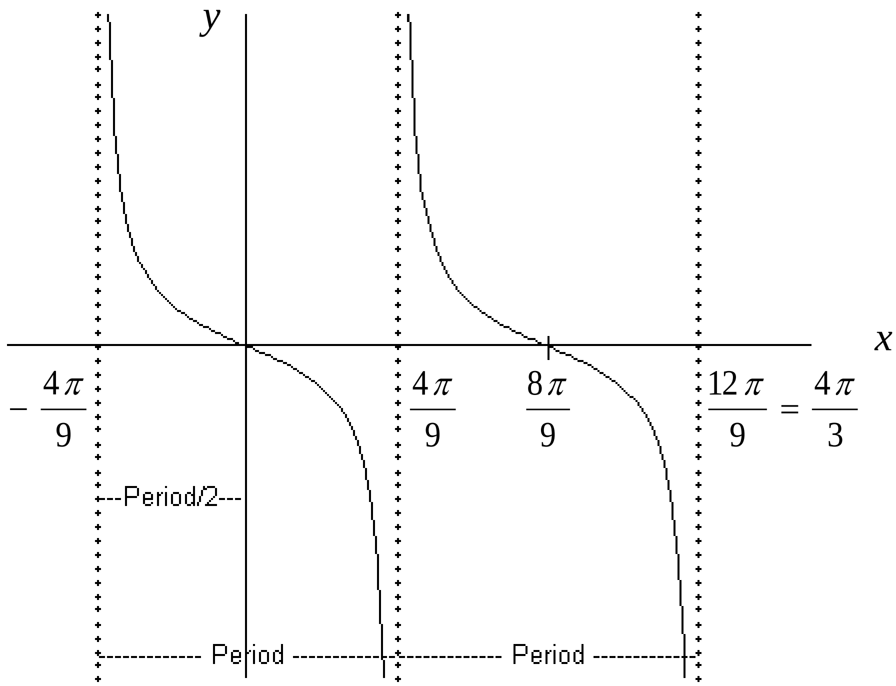
Since the tangent function is being multiplied by a **negative** $\sqrt{21}$, then the graph will be inverted. Thus, we will need to draw two inverted tangent cycles.

Amplitude: None

Period $= \dfrac{\pi}{\frac{9}{8}} = \pi \cdot \dfrac{8}{9} = \dfrac{8\pi}{9}$

Phase Shift: None

$\dfrac{1}{2}$ period $= \dfrac{1}{2} \cdot \dfrac{8\pi}{9} = \dfrac{4\pi}{9}$



The $\frac{4\pi}{3}$ was obtained by $\frac{4\pi}{9} + \frac{8\pi}{9} = \frac{12\pi}{9} = \frac{4\pi}{3}$. That is, we add $\frac{8\pi}{9}$, which is the period, to the starting point of $\frac{4\pi}{9}$, which is where the first vertical asymptote to the right of the *y*-axis crosses the *x*-axis.

The $\frac{8\pi}{9}$ was obtained by $\frac{1}{2}\left(\frac{4\pi}{9} + \frac{12\pi}{9}\right) = \frac{1}{2}\left(\frac{16\pi}{9}\right) = \frac{8\pi}{9}$. That is, we found the midpoint of $\frac{4\pi}{9}$ and $\frac{12\pi}{9}$. Of course, you could also use the fact that two consecutive *x*-intercepts are a distance of the period from each other. Since the first *x*-intercept is 0 and the period is $\frac{8\pi}{9}$, then the second *x*-intercept is $\frac{8\pi}{9}$.

3. $y = \frac{4}{7}\tan(-\pi x)$

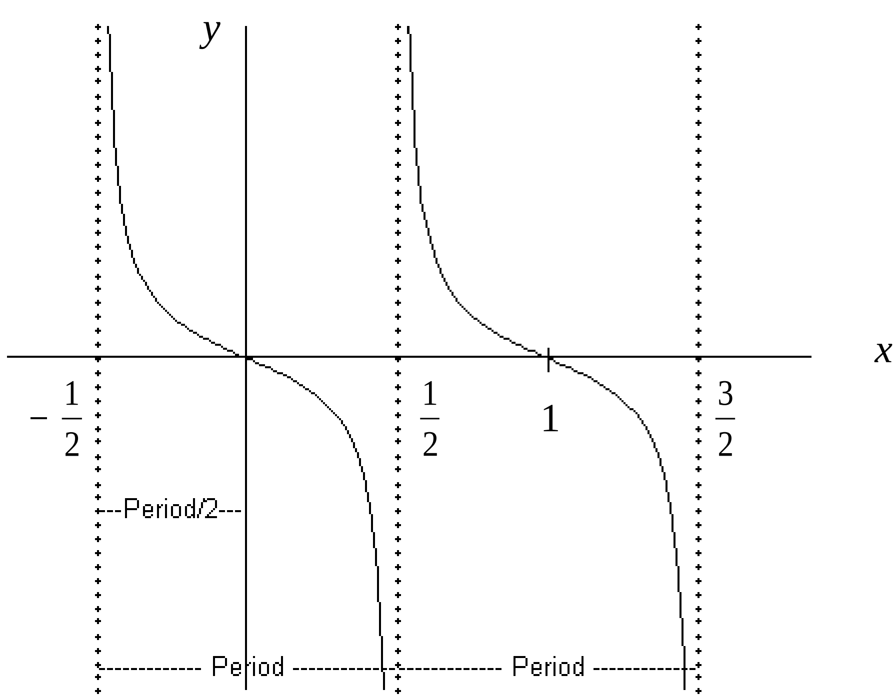
NOTE: Since the tangent function is an odd function, then $\tan(-\pi x) = -\tan \pi x$. Thus, we have that

$$y = \frac{4}{7}\tan(-\pi x) = -\frac{4}{7}\tan \pi x$$

Since the tangent function is being multiplied by a **negative** $\frac{4}{7}$, then the graph will be inverted. Thus, we will need to draw two inverted tangent cycles.

Amplitude: None Period $= \frac{\pi}{\pi} = 1$ Phase Shift: None

$\frac{1}{2}$ period $= \frac{1}{2}$



The $\frac{3}{2}$ was obtained by $\frac{1}{2} + 1 = \frac{1}{2} + \frac{2}{2} = \frac{3}{2}$. That is, we add 1, which is the period, to the starting point of $\frac{1}{2}$, which is where the first vertical asymptote to the right of the *y*-axis crosses the *x*-axis.

The 1 was obtained by $\frac{1}{2}\left(\frac{1}{2} + \frac{3}{2}\right) = \frac{1}{2}\left(\frac{4}{2}\right) = \frac{1}{2}(2) = 1$. That is, we found the midpoint of $\frac{1}{2}$ and $\frac{3}{2}$. Of course, you could also use the fact that two consecutive *x*-intercepts are a distance of the period from each other. Since the first *x*-intercept is 0 and the period is 1, then the second *x*-intercept is 1.

4. $y = 12 \tan\left(3x - \frac{4\pi}{7}\right)$

Since the coefficient of the *x* variable is 3, then we will need to factor the 3 out in order to identify the phase shift. Thus, we have that

$$y = 12 \tan\left(3x - \frac{4\pi}{7}\right) = 12 \tan\left[3\left(x - \frac{4\pi}{21}\right)\right]$$

Amplitude: None

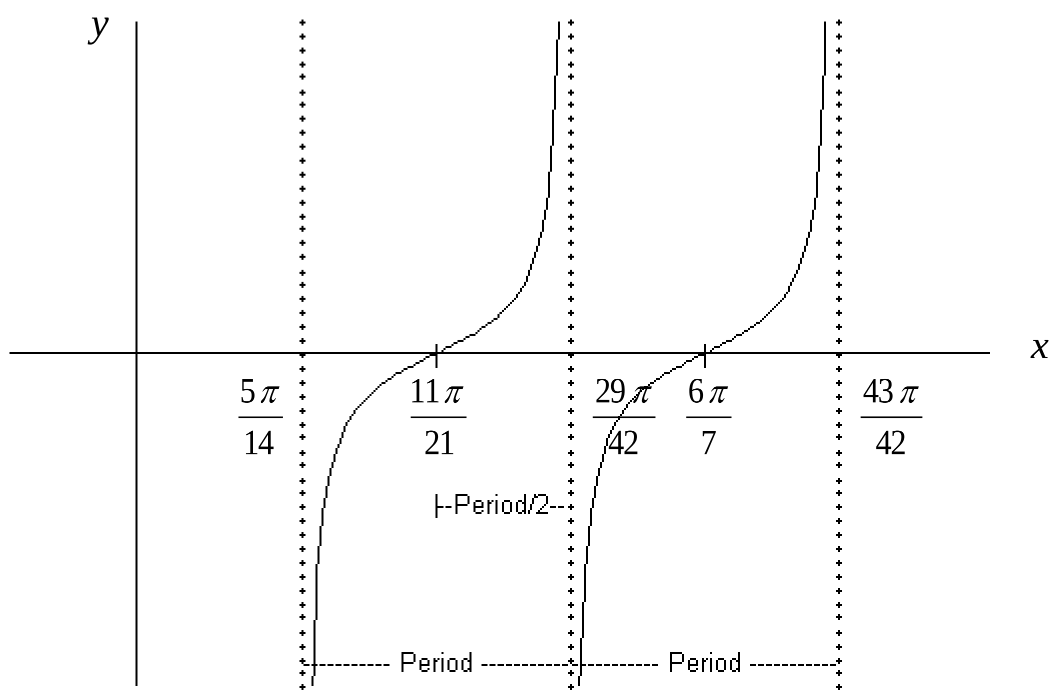
Period $= \frac{\pi}{3}$

Phase Shift: $\frac{4\pi}{21}$ units to the right

$\frac{1}{2}$ period $= \frac{1}{2} \cdot \frac{\pi}{3} = \frac{\pi}{6}$

For the unshifted tangent graph the first two consecutive vertical asymptotes are symmetric about the *y*-axis separated by a distance of the period, which is $\frac{\pi}{3}$ for this function. Thus, the first vertical asymptote to the right of the

y-axis for the **unshifted** tangent graph is $x = \dfrac{\pi}{6}$. Let's shift this vertical asymptote $\dfrac{4\pi}{21}$ units to the right. Thus, the first vertical asymptote for the **shifted** graph is $x = \dfrac{5\pi}{14}$, obtained by $\dfrac{\pi}{6} + \dfrac{4\pi}{21} = \dfrac{7\pi}{42} + \dfrac{8\pi}{42} = \dfrac{15\pi}{42} = \dfrac{5\pi}{14}$. That is, we add $\dfrac{4\pi}{21}$, which is the amount of the shift, to the starting point of $\dfrac{\pi}{6}$, which is the first **unshifted** vertical asymptote to the right of the y-axis crosses the x-axis.



The $\dfrac{29\pi}{42}$ was obtained by $\dfrac{5\pi}{14} + \dfrac{\pi}{3} = \dfrac{15\pi}{42} + \dfrac{14\pi}{42} = \dfrac{29\pi}{42}$. That is, we add $\dfrac{\pi}{3}$, which is the period, to the starting point of $\dfrac{5\pi}{14}$, which is where the first **shifted** vertical asymptote to the right of the y-axis crosses the x-axis.

The $\frac{43\pi}{42}$ was obtained by $\frac{29\pi}{42} + \frac{14\pi}{42} = \frac{43\pi}{42}$. That is, we add $\frac{\pi}{3} = \frac{14\pi}{42}$, which is the period, to the starting point of $\frac{29\pi}{42}$, which is where the second vertical asymptote crosses the *x*-axis.

The $\frac{11\pi}{21}$ was obtained by $\frac{1}{2}\left(\frac{5\pi}{14} + \frac{29\pi}{42}\right) = \frac{1}{2}\left(\frac{15\pi}{42} + \frac{29\pi}{42}\right) =$

$\frac{1}{2}\left(\frac{44\pi}{42}\right) = \frac{1}{2}\left(\frac{22\pi}{21}\right) = \frac{11\pi}{21}$. That is, we found the midpoint of $\frac{5\pi}{14}$ and $\frac{29\pi}{42}$.

The $\frac{6\pi}{7}$ was obtained by $\frac{1}{2}\left(\frac{29\pi}{42} + \frac{43\pi}{42}\right) = \frac{1}{2}\left(\frac{72\pi}{42}\right) = \frac{1}{2}\left(\frac{36\pi}{21}\right) =$

$\frac{1}{2}\left(\frac{12\pi}{7}\right) = \frac{6\pi}{7}$. That is, we found the midpoint of $\frac{29\pi}{42}$ and $\frac{43\pi}{42}$. Of course, you could also use the fact that two consecutive *x*-intercepts are a distance of the period from each other. Since the first *x*-intercept, that we found using the midpoint formula, is $\frac{11\pi}{21}$ and the period is $\frac{\pi}{3} = \frac{7\pi}{21}$, then the second *x*-intercept is $\frac{11\pi}{21} + \frac{7\pi}{21} = \frac{18\pi}{21} = \frac{6\pi}{7}$.

5. $y = \tan\left(\frac{6x}{7} + \frac{11\pi}{21}\right)$

Since the coefficient of the *x* variable is $\frac{6}{7}$, then we will need to factor the $\frac{6}{7}$ out in order to identify the phase shift. Thus, we have that

$$y = \tan\left(\frac{6x}{7} + \frac{11\pi}{21}\right) = \tan\left[\frac{6}{7}\left(x + \frac{11\pi}{18}\right)\right]$$

NOTE: The $\frac{11\pi}{18}$ was obtained by $\frac{11\pi}{21} \div \frac{6}{7} = \frac{11\pi}{21} \cdot \frac{7}{6} = \frac{11\pi}{3} \cdot \frac{1}{6} = \frac{11\pi}{18}$.

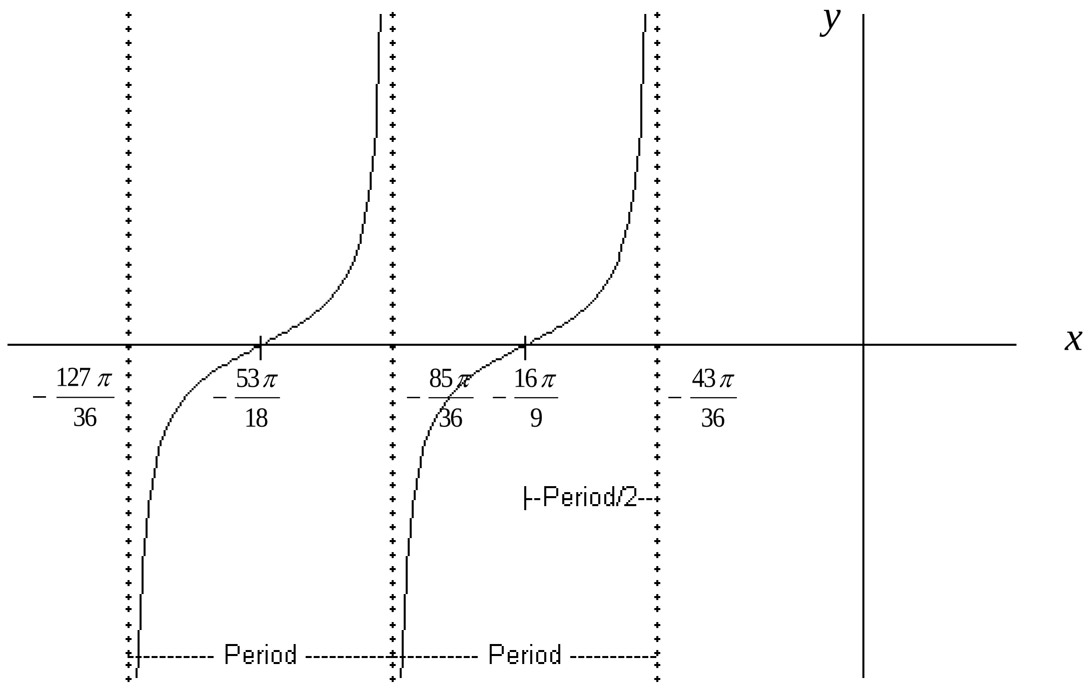
Amplitude: None

$$\text{Period} = \frac{\pi}{\frac{6}{7}} = \pi \cdot \frac{7}{6} = \frac{7\pi}{6}$$

Phase Shift: $\frac{11\pi}{18}$ units to the left

$$\frac{1}{2}\text{period} = \frac{1}{2} \cdot \frac{7\pi}{6} = \frac{7\pi}{12}$$

For the unshifted tangent graph the first two consecutive vertical asymptotes are symmetric about the *y*-axis separated by a distance of the period, which is $\frac{7\pi}{6}$ for this function. Thus, the first vertical asymptote to the left of the *y*-axis for the **unshifted** tangent graph is $x = -\frac{7\pi}{12}$. Let's shift this vertical asymptote $\frac{11\pi}{18}$ units to the left. Thus, the first vertical asymptote for the **shifted** graph is $x = -\frac{43\pi}{36}$, obtained by $-\frac{7\pi}{12} - \frac{11\pi}{18} = -\frac{21\pi}{36} - \frac{22\pi}{36} = -\frac{43\pi}{36}$. That is, we subtract $\frac{11\pi}{18}$, which is the amount of the shift, to the starting point of $-\frac{7\pi}{12}$, which is the first **unshifted** vertical asymptote to the right of the *y*-axis crosses the *x*-axis. Note that we subtract the amount of the shift because we are moving to the left.



The $-\dfrac{85\pi}{36}$ was obtained by $-\dfrac{43\pi}{36} - \dfrac{7\pi}{6} = -\dfrac{43\pi}{36} - \dfrac{42\pi}{36} = -\dfrac{85\pi}{36}$. That is, we subtract $\dfrac{7\pi}{6}$, which is the period, to the starting point of $-\dfrac{43\pi}{36}$, which is where the first **shifted** vertical asymptote to the left of the *y*-axis crosses the *x*-axis.

The $-\dfrac{127\pi}{36}$ was obtained by $-\dfrac{85\pi}{36} - \dfrac{42\pi}{36} = -\dfrac{127\pi}{36}$. That is, we subtract $\dfrac{7\pi}{6} = \dfrac{42\pi}{36}$, which is the period, to the starting point of $-\dfrac{85\pi}{36}$, which is where the second vertical asymptote crosses the *x*-axis.

The $-\dfrac{16\pi}{9}$ was obtained by $\dfrac{1}{2}\left[-\dfrac{85\pi}{36} + \left(-\dfrac{43\pi}{36}\right)\right] = \dfrac{1}{2}\left(-\dfrac{128\pi}{36}\right) =$

$\dfrac{1}{2}\left(-\dfrac{32\pi}{9}\right) = -\dfrac{16\pi}{9}$. That is, we found the midpoint of $-\dfrac{85\pi}{36}$ and $-\dfrac{43\pi}{36}$.

The $-\dfrac{53\pi}{18}$ was obtained by $\dfrac{1}{2}\left[-\dfrac{127\pi}{36}+\left(-\dfrac{85\pi}{36}\right)\right]=\dfrac{1}{2}\left(-\dfrac{212\pi}{36}\right)=$ $\dfrac{1}{2}\left(-\dfrac{53\pi}{9}\right)=-\dfrac{53\pi}{18}$. That is, we found the midpoint of $-\dfrac{127\pi}{36}$ and $-\dfrac{85\pi}{36}$. Of course, you could also use the fact that two consecutive *x*-intercepts are a distance of the period from each other. Since the first *x*-intercept, that we found using the midpoint formula, is $-\dfrac{16\pi}{9}$ and the period is $\dfrac{7\pi}{6}$, then the second *x*-intercept is $-\dfrac{16\pi}{9}-\dfrac{7\pi}{6}=-\dfrac{32\pi}{18}-\dfrac{21\pi}{18}=$ $-\dfrac{53\pi}{18}$.

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6. COTANGENT GRAPHS

Example Find the *x*-intercepts of the graph of $y=\cot x$ in the interval $[-2\pi, 2\pi]$.

NOTE: The interval $[-2\pi, 2\pi]$ of angles are the angles going one time around the Unit Circle clockwise for the subinterval $[-2\pi, 0]$ and the angles going one time around the Unit Circle counterclockwise for the subinterval $[0, 2\pi]$.

To find the *x*-intercepts, set y equal to 0: $\cot x = 0$.

Since $\cot x = \dfrac{\cos x}{\sin x}$, then $\cot x = 0 \Rightarrow \dfrac{\cos x}{\sin x} = 0$.

Since a fraction can only equal zero when the numerator of the fraction equals zero, then $\frac{\cos x}{\sin x} = 0 \Rightarrow \cos x = 0$. By Unit Circle Trigonometry, we are looking for angles in the interval $[-2\pi, 2\pi]$ that intersect the Unit Circle so that the *x*-coordinate of the point of intersection is 0. Going around the Unit Circle clockwise, these angles are $x = -\frac{\pi}{2}, -\frac{3\pi}{2}$. Going around the Unit Circle counterclockwise, these angles are $x = \frac{\pi}{2}, \frac{3\pi}{2}$.

Thus, the *x*-intercepts of the graph of $y = \cot x$ in the interval $[-2\pi, 2\pi]$ are the points $\left(-\frac{3\pi}{2}, 0\right)$, $\left(-\frac{\pi}{2}, 0\right)$, $\left(\frac{\pi}{2}, 0\right)$, and $\left(\frac{3\pi}{2}, 0\right)$.

Example Find the vertical asymptotes of the graph of $y = \cot x$ in the interval $[-2\pi, 2\pi]$.

NOTE: The interval $[-2\pi, 2\pi]$ of angles are the angles going one time around the Unit Circle clockwise for the subinterval $[-2\pi, 0]$ and the angles going one time around the Unit Circle counterclockwise for the subinterval $[0, 2\pi]$.

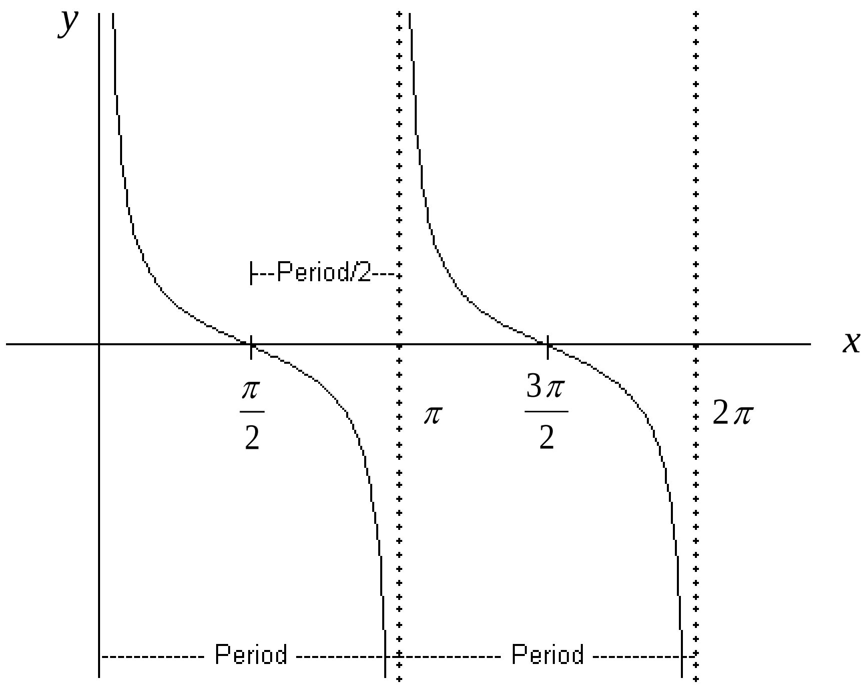
Since $\cot x = \frac{\cos x}{\sin x}$, then the vertical asymptotes will occur where the denominator of this fraction is equal to zero. Thus, we want to solve the equation $\sin x = 0$ in the interval $[-2\pi, 2\pi]$.

By Unit Circle Trigonometry, we are looking for angles in the interval $[-2\pi, 2\pi]$ that intersect the Unit Circle so that the *y*-coordinate of the point of intersection is 0. Going around the Unit Circle clockwise, these angles are $x = 0, -\pi, -2\pi$. Going around the Unit Circle counterclockwise, these angles are $x = 0, \pi, 2\pi$.

Thus, the vertical asymptotes of the graph of $y = \cot x$ in the interval $[-2\pi, 2\pi]$ are $x = -2\pi$, $x = -\pi$, $x = 0$, $x = \pi$, and $x = 2\pi$.

NOTE: The vertical line given by the equation $x = 0$ is the y-axis.

Example Sketch two cycles of the graph of $y = \cot x$ using the x-intercepts and vertical asymptotes of the function.



The function $y = \cot x$ does not have an amplitude and the period of the function is π.

In general, given the function $y = a\cot(bx + c)$, we may write this function as $y = a\cot\left[b\left(x + \frac{c}{b}\right)\right]$ by factoring out b. The cotangent function does not have an amplitude, the period is $\frac{\pi}{|b|}$, and the phase shift is $\frac{c}{b}$ units to the right if $\frac{c}{b} < 0$ or is $\frac{c}{b}$ units to the left if $\frac{c}{b} > 0$.

Theorem The cotangent function is an odd function. That is, $\cot(-\theta) = -\cot\theta$ for all θ in the domain of the function.

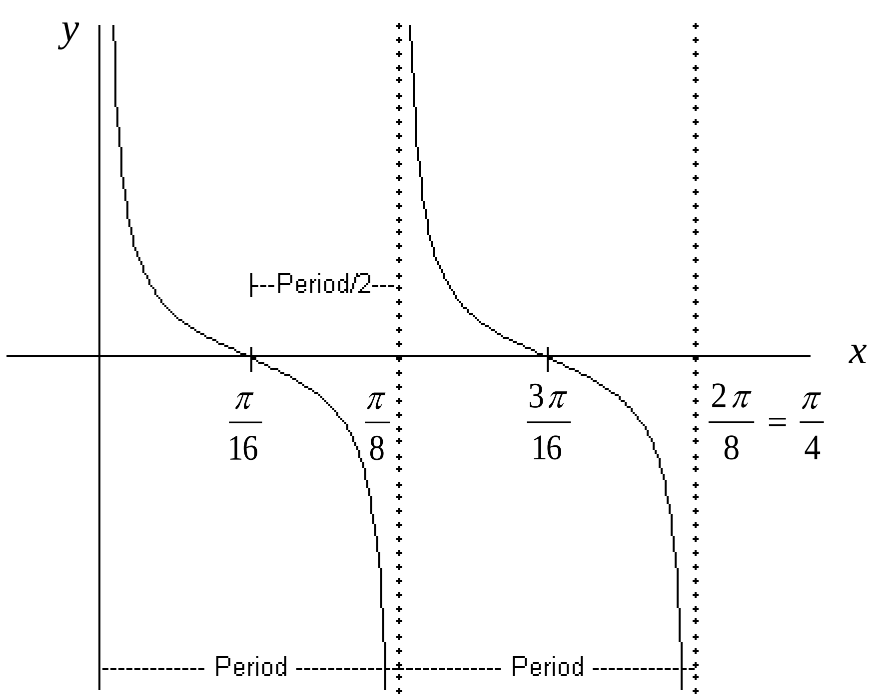
Like the tangent function, a sketch of the graph of the cotangent function can be obtained from the x-intercepts and vertical asymptotes of the function. Two consecutive vertical asymptotes are a distance of the **period** from each other. The x-coordinates of the x-intercepts are midway between two consecutive vertical asymptotes. That is, the x-coordinates of the x-intercepts are the midpoint of two consecutive vertical asymptotes.

So to sketch the graph of a cotangent function, you only need to know where the first vertical asymptote is located. Then use the period to find the next consecutive vertical asymptote. For an unshifted cotangent function, the first vertical asymptote is the ***y*-axis**. For a shifted cotangent function, first shift one vertical asymptote for the unshifted graph. Once you have the vertical asymptotes, you find the x-coordinates of the x-intercepts by finding the midpoint of two consecutive vertical asymptotes.

Examples Sketch two cycles of the graph of the following functions.

1. $y = \sqrt{7}\cot 8x$

Amplitude: None Period $= \dfrac{\pi}{8}$ Phase Shift: None



The $\frac{\pi}{16}$ was obtained by $\frac{1}{2}\left(0 + \frac{\pi}{8}\right) = \frac{1}{2}\left(\frac{\pi}{8}\right) = \frac{\pi}{16}$. That is, we found the midpoint of 0 and $\frac{\pi}{8}$.

The $\frac{3\pi}{16}$ was obtained by $\frac{1}{2}\left(\frac{\pi}{8} + \frac{2\pi}{8}\right) = \frac{1}{2}\left(\frac{3\pi}{8}\right) = \frac{3\pi}{16}$. That is, we found the midpoint of $\frac{\pi}{8}$ and $\frac{\pi}{4} = \frac{2\pi}{8}$. Of course, you could also use the fact that two consecutive *x*-intercepts are a distance of the period from each other. Since the first *x*-intercept, that we found using the midpoint formula, is $\frac{\pi}{16}$ and the period is $\frac{\pi}{8} = \frac{2\pi}{16}$, then the second *x*-intercept is $\frac{\pi}{16} + \frac{2\pi}{16} = \frac{3\pi}{16}$.

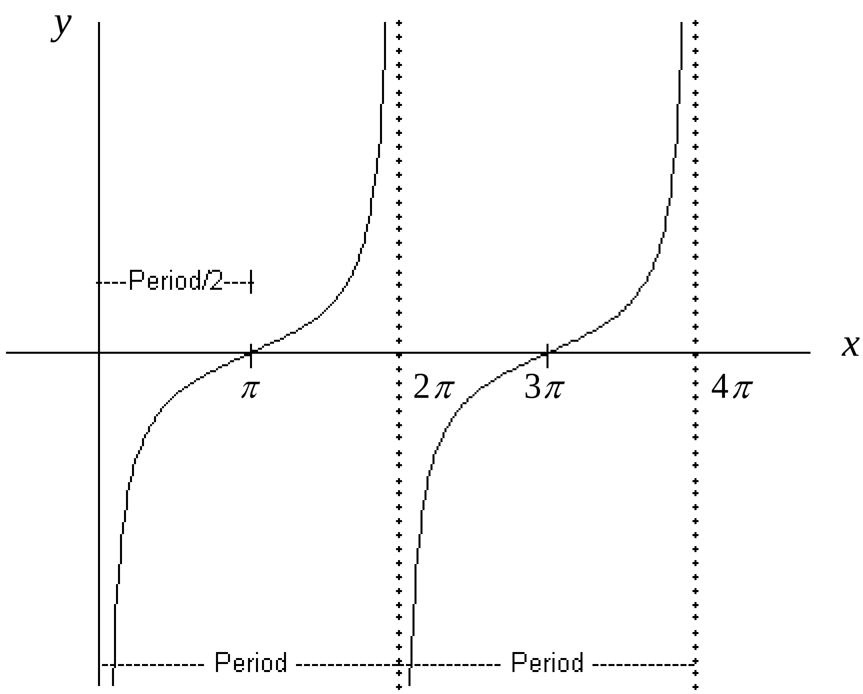
2. $y = 15 \cot\left(-\frac{x}{2}\right)$

NOTE: Since the cotangent function is an odd function, then $\cot\left(-\frac{x}{2}\right) = -\cot\left(\frac{1}{2}x\right)$. Thus, we have that

$$y = 15 \cot\left(-\frac{x}{2}\right) = -15 \cot\left(\frac{1}{2}x\right)$$

Since the cotangent function is being multiplied by a **negative** 15, then the graph will be inverted. Thus, we will need to draw two inverted cotangent cycles.

Amplitude: None   Period $= \dfrac{\pi}{\frac{1}{2}} = 2\pi$ Phase Shift: None



The π was obtained by $\frac{1}{2}(0 + 2\pi) = \frac{1}{2}(2\pi) = \pi$. That is, we found the midpoint of 0 and π.

The 3π was obtained by $\frac{1}{2}(2\pi + 4\pi) = \frac{1}{2}(6\pi) = 3\pi$. That is, we found the midpoint of 2π and 4π. Of course, you could also use the fact that two consecutive *x*-intercepts are a distance of the period from each other. Since the first *x*-intercept, that we found using the midpoint formula, is π and the period is 2π, then the second *x*-intercept is $\pi + 2\pi = 3\pi$.

3. $y = \dfrac{3}{5}\cot\left(\dfrac{9\pi}{2} - 6x\right)$

Since the coefficient of the x variable is -6, then we will need to factor the -6 out in order to identify the phase shift. Thus, we have that

$$y = \frac{3}{5}\cot\left(\frac{9\pi}{2} - 6x\right) = \frac{3}{5}\cot\left(-6x + \frac{9\pi}{2}\right) = \frac{3}{5}\cot\left[-6\left(x - \frac{3\pi}{4}\right)\right]$$

NOTE: The $\frac{3\pi}{4}$ was obtained by $\frac{9\pi}{2} \div 6 = \frac{9\pi}{2}\cdot\frac{1}{6} = \frac{3\pi}{2}\cdot\frac{1}{2} = \frac{3\pi}{4}$.

NOTE: Since the cotangent function is an odd function, then $\cot\left[-6\left(x - \frac{3\pi}{4}\right)\right] = -\cot\left[6\left(x - \frac{3\pi}{4}\right)\right]$. Thus, we have that

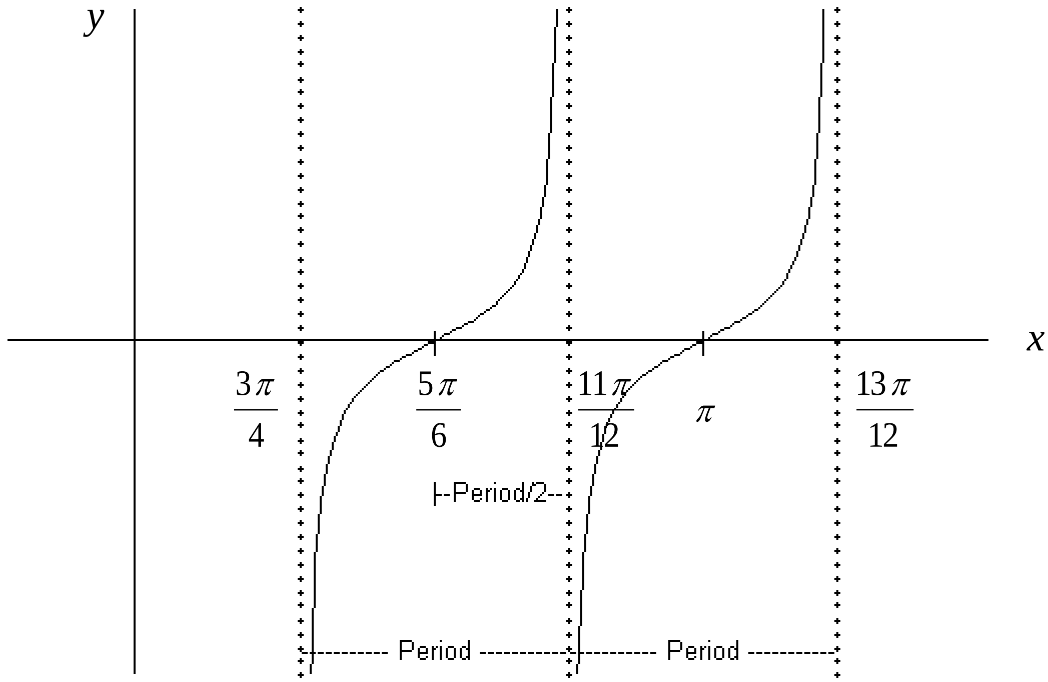
$$y = \frac{3}{5}\cot\left[-6\left(x - \frac{3\pi}{4}\right)\right] = -\frac{3}{5}\cot\left[6\left(x - \frac{3\pi}{4}\right)\right]$$

Since the cotangent function is being multiplied by a **negative** $\frac{3}{5}$, then the graph will be inverted. Thus, we will need to draw two inverted cotangent cycles.

Amplitude: None Period $= \frac{\pi}{6}$

Phase Shift: $\frac{3\pi}{4}$ units to the right

For the unshifted cotangent graph the first vertical asymptote is the y-axis. Let's shift this vertical asymptote $\frac{3\pi}{4}$ units to the right. Thus, the first vertical asymptote for the **shifted** graph is $x = \frac{3\pi}{4}$, obtained by $0 + \frac{3\pi}{4} = \frac{3\pi}{4}$. That is, we add $\frac{3\pi}{4}$, which is the amount of the shift, to the starting point of 0.



The $\frac{11\pi}{12}$ was obtained by $\frac{3\pi}{4} + \frac{\pi}{6} = \frac{9\pi}{12} + \frac{2\pi}{12} = \frac{11\pi}{12}$. That is, we add $\frac{\pi}{6}$, which is the period, to the starting point of $\frac{3\pi}{4}$, which is where the first **shifted** vertical asymptote to the right of the *y*-axis crosses the *x*-axis.

The $\frac{13\pi}{12}$ was obtained by $\frac{11\pi}{12} + \frac{2\pi}{12} = \frac{13\pi}{12}$. That is, we add $\frac{\pi}{6} = \frac{2\pi}{12}$, which is the period, to the starting point of $\frac{11\pi}{12}$, which is where the second vertical asymptote crosses the *x*-axis.

The $\frac{5\pi}{6}$ was obtained by $\frac{1}{2}\left(\frac{3\pi}{4} + \frac{11\pi}{12}\right) = \frac{1}{2}\left(\frac{9\pi}{12} + \frac{11\pi}{12}\right) =$

$\frac{1}{2}\left(\frac{20\pi}{12}\right) = \frac{1}{2}\left(\frac{5\pi}{3}\right) = \frac{5\pi}{6}$. That is, we found the midpoint of $\frac{3\pi}{4}$ and $\frac{11\pi}{12}$.

The π was obtained by $\frac{1}{2}\left(\frac{11\pi}{12} + \frac{13\pi}{12}\right) = \frac{1}{2}\left(\frac{24\pi}{12}\right) = \frac{1}{2}(2\pi) = \pi$. That is, we found the midpoint of $\frac{11\pi}{12}$ and $\frac{13\pi}{12}$. Of course, you could also use the fact that two consecutive *x*-intercepts are a distance of the period from each other. Since the first *x*-intercept, that we found using the midpoint formula, is $\frac{5\pi}{6}$ and the period is $\frac{\pi}{6}$, then the second *x*-intercept is $\frac{5\pi}{6} + \frac{\pi}{6} = \frac{6\pi}{6} = \pi$.

4. $y = 2\cot\left(\frac{x}{7} + 4\pi\right)$

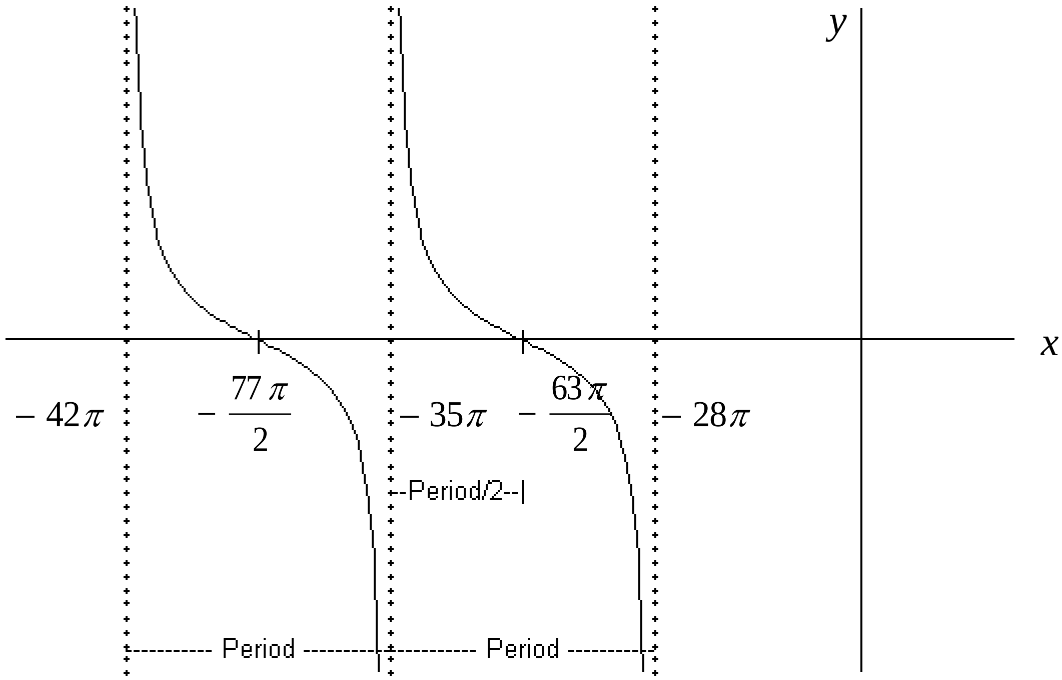
Since the coefficient of the *x* variable is $\frac{1}{7}$, then we will need to factor the $\frac{1}{7}$ out in order to identify the phase shift. Thus, we have that

$$y = 2\cot\left(\frac{x}{7} + 4\pi\right) = 2\cot\left[\frac{1}{7}(x + 28\pi)\right]$$

Amplitude: None Period $= \dfrac{\pi}{\frac{1}{7}} = 7\pi$

Phase Shift: 28π units to the left

For the unshifted cotangent graph the first vertical asymptote is the *y*-axis. Let's shift this vertical asymptote 28π units to the right. Thus, the first vertical asymptote for the **shifted** graph is $x = -28\pi$, obtained by $0 - 28\pi = -28\pi$. That is, we subtract 28π, which is the amount of the shift, to the starting point of 0. Note that we subtract the amount of the shift because we are moving to the left.



The -35π was obtained by $-28\pi - 7\pi = -35\pi$. That is, we subtract 7π, which is the period, to the starting point of -28π, which is where the first **shifted** vertical asymptote to the left of the *y*-axis crosses the *x*-axis.

The -42π was obtained by $-35\pi - 7\pi = -42\pi$. That is, we subtract 7π, which is the period, to the starting point of -35π, which is where the second **shifted** vertical asymptote to the left of the *y*-axis crosses the *x*-axis.

The $-\dfrac{63\pi}{2}$ was obtained by $\dfrac{1}{2}[-35\pi + (-28\pi)] = \dfrac{1}{2}(-63\pi) = -\dfrac{63\pi}{2}$.

That is, we found the midpoint of -35π and -28π.

The $-\dfrac{77\pi}{2}$ was obtained by $\dfrac{1}{2}[-42\pi + (-35\pi)] = \dfrac{1}{2}(-77\pi) = -\dfrac{77\pi}{2}$.

That is, we found the midpoint of -42π and -35π. Of course, you could also use the fact that two consecutive *x*-intercepts are a distance of the period from each other. Since the first *x*-intercept, that we found using the midpoint formula, is $-\dfrac{63\pi}{2}$ and the period is $7\pi = \dfrac{14\pi}{2}$, then the second *x*-intercept is $-\dfrac{63\pi}{2} - \dfrac{14\pi}{2} = -\dfrac{77\pi}{2}$.

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