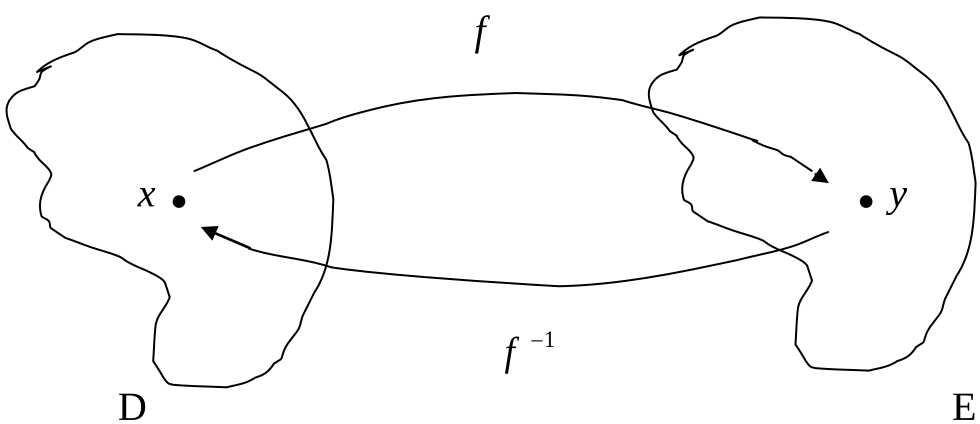
LESSON 9 THE INVERSE TRIGONOMETRIC FUNCTIONS

Topics in this lesson:

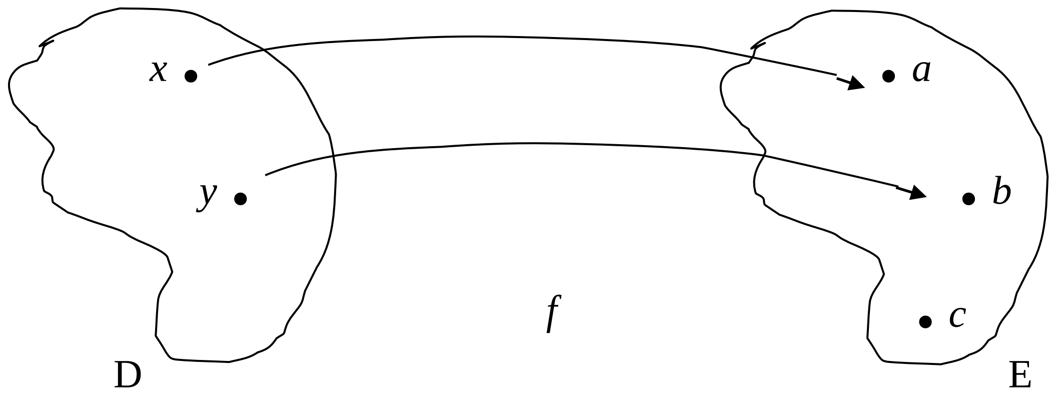
1. REVIEW OF INVERSE FUNCTIONS

Example Consider the following function f and its inverse f^{-1}.



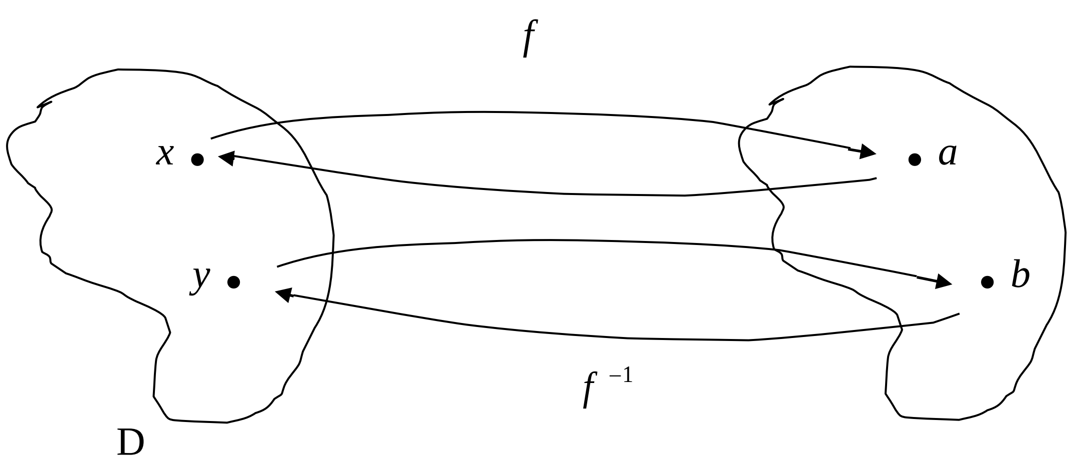
NOTE: The function f maps x in the set D to y in the set E and f^{-1} maps y back to x. Of course, this is what an inverse function is suppose to do.

Example Consider the following function f. Does f have an inverse?



The function f maps x in the set D to a in the set E. So, the inverse function would map a back to x. The function f maps y in the set D to b in the set E. So, the inverse function would map b back to y. However, where does the inverse function map c. Note that the domain of the function f is the set $\{x, y\}$ and the range of the function f is the set $\{a, b\}$.

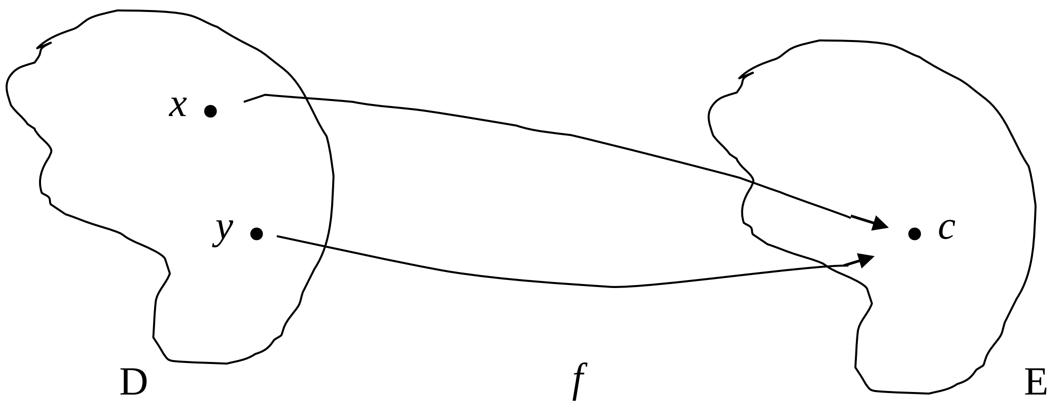
The function f is not an onto function. In order for a function to be an onto function, every element in the set E must be used. For the given function f, the element c in the set E was not used. **In order for a function to have an inverse, it must be an onto function.** If a function is not an onto function, then the lack of this needed condition is easy to fix. To fix the lack of the onto condition, replace the set E by the range of the function.



Range of $f = \{a, b\}$

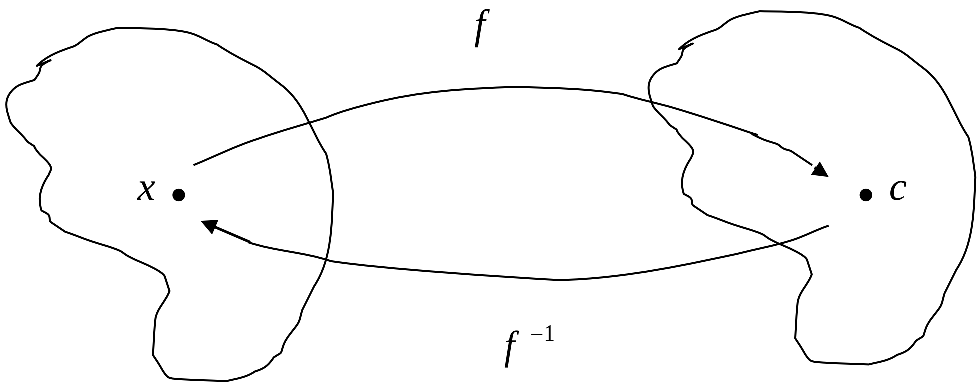
Now, this function has an inverse function. Notice the following relationship above: The domain of the inverse function f^{-1} is equal to the range of the function f.

Example Consider the following function f. Does f have an inverse?



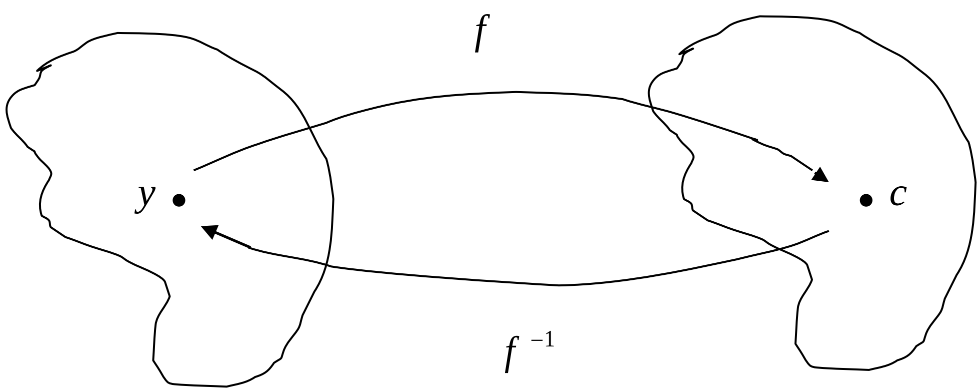
Note that the domain of the function f is the set D, which is the set $\{x, y\}$ and the range of the function f is the set $\{c\}$. Also, note that the function f is an onto function. The set E is the range of the function f. The function f maps *x* in the set D to *c* in the set E. The function f also maps *y* in the set D to *c* in the set E. So, the inverse function would map *c* back to either *x* or *y*. Which one do you use? The problem here is that the function f is not a one-to-one function. In order for a function to be a one-to-one function, you may only use each element in the set E once. **In order for a function to have an inverse, it must be a one-to-one function.** If a function is not a one-to-one function, then the lack of this needed condition is not as easy to fix as the lack of the onto condition. In order to fix the lack of the one-to-one condition, you must put a restriction on the domain of the function. In other words, you must eliminate elements from the set D. What elements in the set D are you going to chose to eliminate? This is the reason that fixing the lack of the one-to-one condition is harder. For the function f, the domain is the set D = $\{x, y\}$. Thus, we will either eliminate *x* or *y*. Each restricted domain will produce an inverse function. Thus, these two choices for the restricted domain will produce two inverse functions.

If we eliminate *y*, then we get the following inverse function:



Restricted domain of $f = \{x\}$ Range of $f = \{c\}$

If we eliminate *x*, then we get the following inverse function:



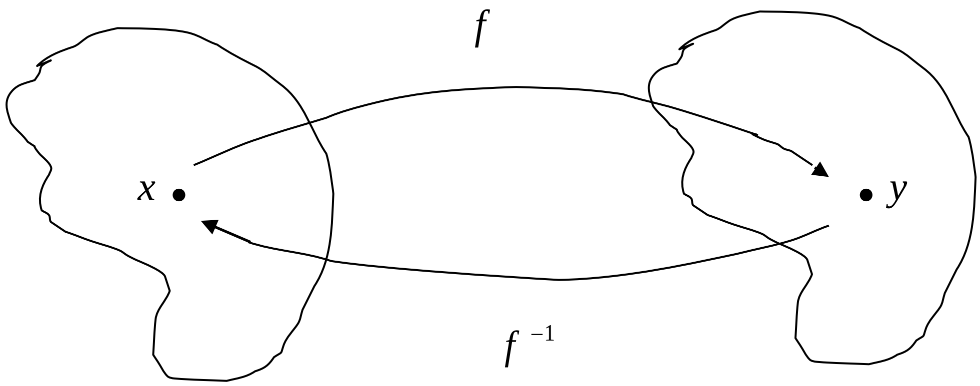
Restricted domain of $f = \{y\}$ Range of $f = \{c\}$

The function f has two possible inverse functions depending on the restricted domain that is chosen.

Notice the following relationship for both inverse functions above: The restricted domain of the function f is equal to the range of the function f^{-1}.

<u>Theorem</u> A function f has an inverse function, denoted by f^{-1} if and only if the function f is one-to-one and onto.

We have the following relationships:



Restricted domain of f || Range of f^{-1}

Range of f || Domain of f^{-1}

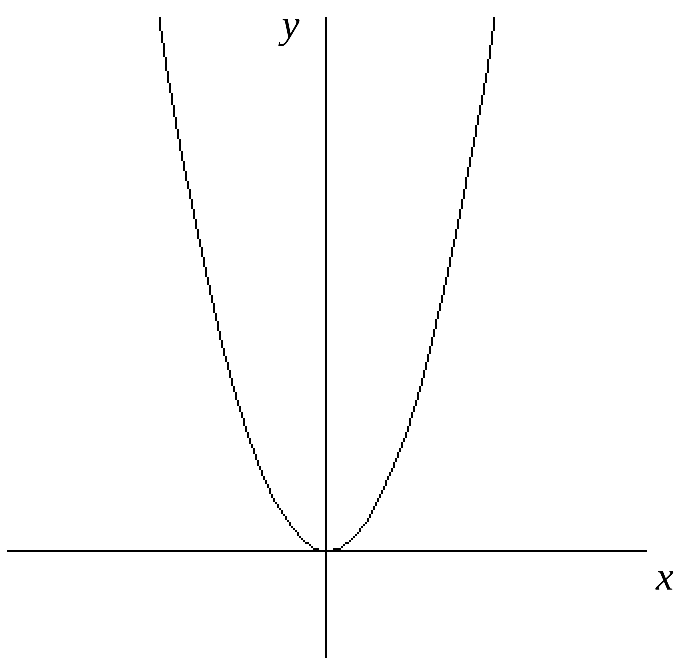
We also have the following two relationships between a function f and its inverse function f^{-1}:

1. $f^{-1}(f(x)) = x$ for all x in the **restricted domain of** f

2. $f(f^{-1}(y)) = y$ for all y in the **domain of** f^{-1}

Example Find the inverse function of the function $f(x) = x^2$.

First, consider the graph of $f(x) = x^2$.



Information about the domain of the function f can be determined by the *x*-coordinate of the points on the graph. Since the value of the *x*-coordinates range in value from negative infinity to positive infinity, then the domain of f is all real numbers. Information about the range of the function f can be determined by the *y*-coordinate of the points on the graph. Since the value of the *y*-coordinates range in value from zero to positive infinity, then the range of f is all real numbers greater than or equal to zero. In interval notation, we have the following:

$$\text{Domain of } f = (-\infty, \infty)$$

$$\text{Range of } f = [0, \infty)$$

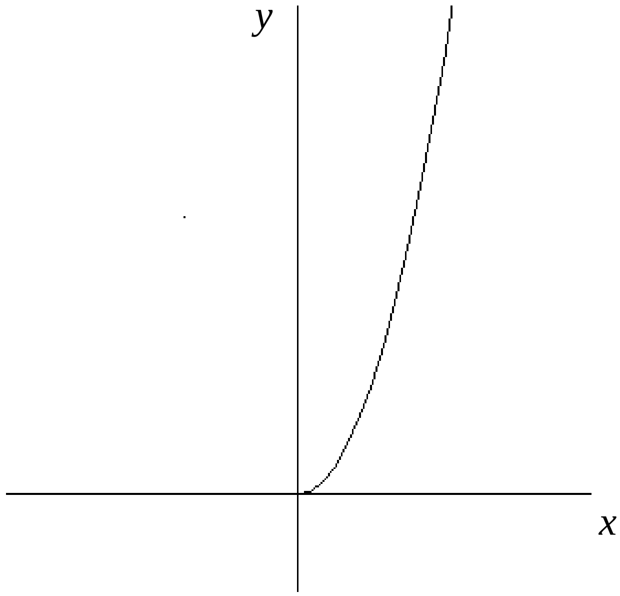
Since the range of f is $[0, \infty)$, then by the discussion above, the domain of f^{-1} is $[0, \infty)$.

Recall the following test for checking the graph of a function for being one-to-one:

The Horizontal Line Test: If a horizontal line intersects the graph of a function in more than one place, then the function is not one-to-one.

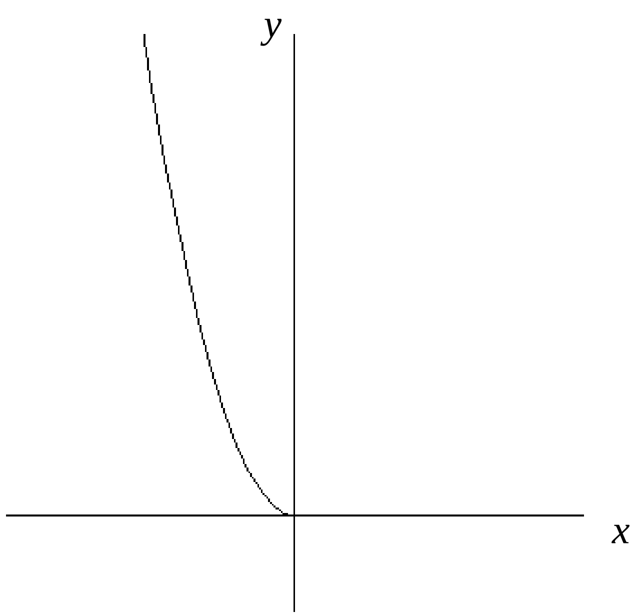
By the horizontal line test, the function f is not one-to-one. We will have to put a restriction on the domain of the function in order to fix this. We have the following two choices for the restricted domain: 1) the interval of numbers $[0, \infty)$; that is, the set of numbers greater than or equal to zero **or** 2) the interval of numbers $(-\infty, 0]$; that is, the set of numbers less than or equal to zero. Each restricted domain will produce an inverse function. Thus, these two choices for the restricted domain will produce two inverse functions.

For the first choice of $[0, \infty)$, our restricted domain is $[0, \infty)$ and the graph of the function f on the restricted domain looks like the following:



Since the restricted domain of f is $[0, \infty)$, then by the discussion above, the range of f^{-1} is $[0, \infty)$.

For the second choice of $(-\infty, 0]$, our restricted domain is $(-\infty, 0]$ and the graph of the function f on the restricted domain looks like the following:



Since the restricted domain of f is $(-\infty, 0]$, then by the discussion above, the range of f^{-1} is $(-\infty, 0]$.

Now, let's find the inverse function(s) algebraically.

Set $f(x) = y$: $\quad y = x^2$

Solve for x: $\quad y = x^2 \Rightarrow \sqrt{y} = \sqrt{x^2} \Rightarrow \sqrt{y} = |x| \Rightarrow x = \pm\sqrt{y}$

Thus, either $f^{-1}(y) = \sqrt{y}$ if the restricted domain of f is $[0, \infty)$

or $\quad f^{-1}(y) = -\sqrt{y}$ if the restricted domain of f is $(-\infty, 0]$

Notice, as predicted above the domain of f^{-1} is $[0, \infty)$ and the range of f^{-1} is $[0, \infty)$ for the first inverse function and the range of f^{-1} is $(-\infty, 0]$ for the second inverse function.

Now, let's verify the two relationships for this function and its two inverse functions are true.

1. For the first restricted of domain of $[0, \infty)$ for the function $f(x) = x^2$, we have that $f^{-1}(y) = \sqrt{y}$:

   a. $f^{-1}(f(x)) = f^{-1}(x^2) = \sqrt{x^2} = |x| = x$. Note that since x belongs to the restricted domain of $[0, \infty)$, then x is a positive number. The absolute value of a positive number is itself.

   b. $f(f^{-1}(y)) = f(\sqrt{y}) = (\sqrt{y})^2 = y$. Note that since y is in the domain of f^{-1} and the domain of f^{-1} is $[0, \infty)$, then y is greater than or equal to zero. Thus, the square root of y is defined.

2. For the second restricted of domain of $(-\infty, 0]$ for the function $f(x) = x^2$, we have that $f^{-1}(y) = -\sqrt{y}$:

a. $f^{-1}(f(x)) = f^{-1}(x^2) = -\sqrt{x^2} = -|x| = -(-x) = x$. Note that since x belongs to the restricted domain of $(-\infty, 0]$, then x is a negative number. The absolute value of a negative number is the negative of itself. Thus, $|x| = -x$.

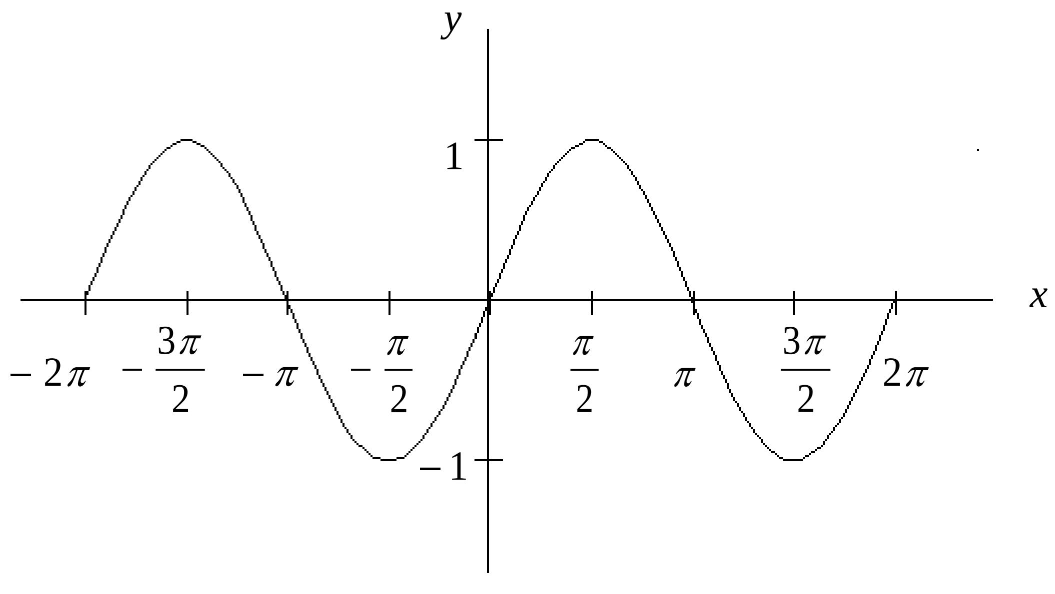
b. $f(f^{-1}(y)) = f(-\sqrt{y}) = (-\sqrt{y})^2 = y$. Note that since y is in the domain of f^{-1} and the domain of f^{-1} is $[0, \infty)$, then y is greater than or equal to zero. Thus, the square root of y is defined.

Thus, the two relationships for this function and its two inverse functions hold.

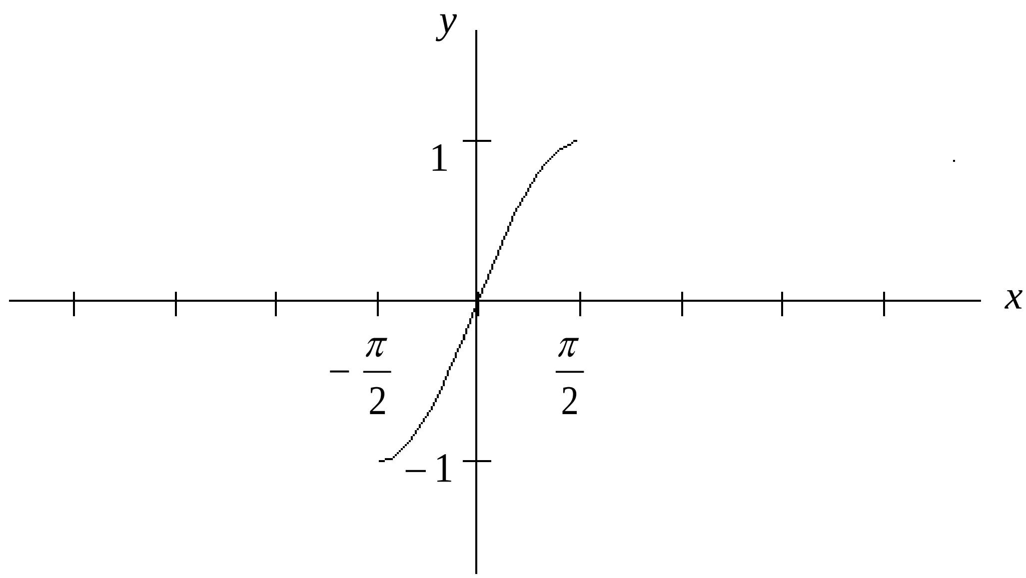
Back to Topics List

2. THE INVERSE SINE FUNCTION

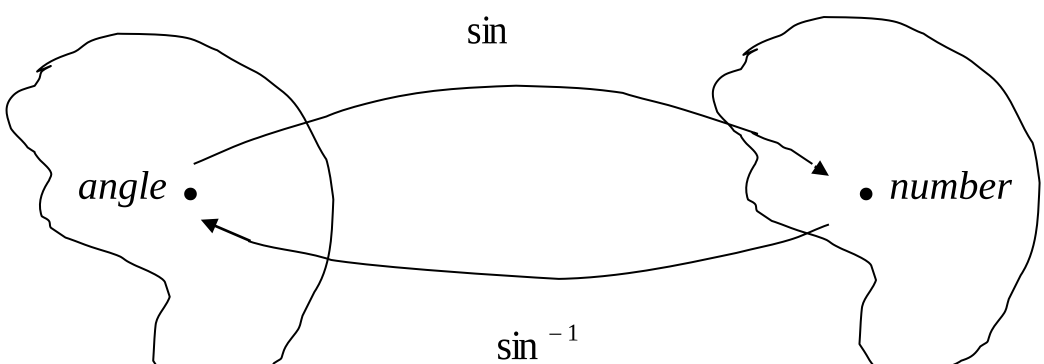
Consider the following two cycles of the graph of $y = \sin x$:



The range of $y = \sin x$ is the interval $[-1, 1]$. Thus, the domain of the inverse sine function is $[-1, 1]$. The domain of $y = \sin x$ is the set of all real numbers. However, by the horizontal line test, the function $y = \sin x$ is not one-to-one. Thus, we will need to put a restriction of the domain. We will restrict the domain of the function $y = \sin x$ to the interval of angles $\left[-\frac{\pi}{2}, \frac{\pi}{2} \right]$. The graph of $y = \sin x$ on the restricted domain looks like the following:



Since the restricted domain of $y = \sin x$ is the interval of angles $\left[-\frac{\pi}{2}, \frac{\pi}{2} \right]$, then range of the inverse sine function is the interval of angles $\left[-\frac{\pi}{2}, \frac{\pi}{2} \right]$.



Restricted domain of $\sin = \left[-\frac{\pi}{2}, \frac{\pi}{2} \right]$ Range of $\sin = [-1, 1]$

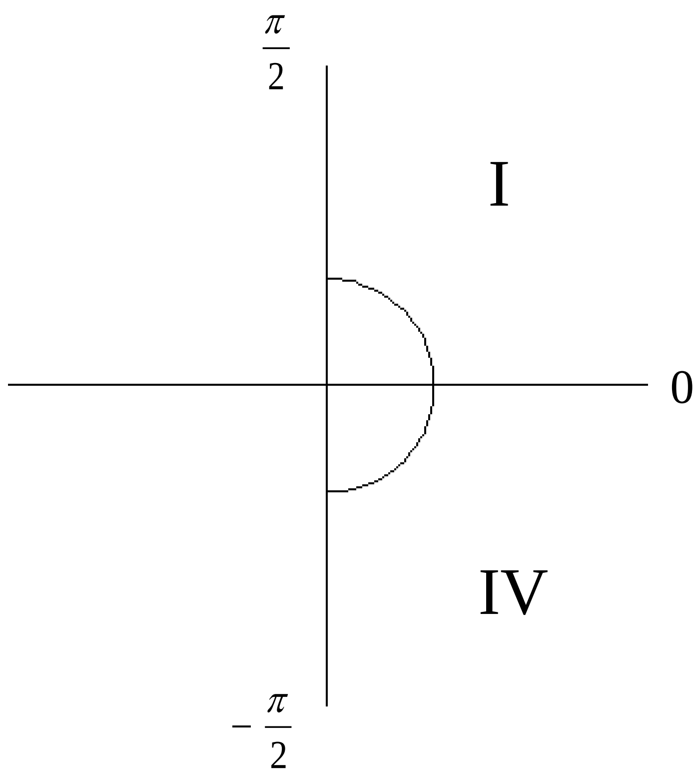
|| ||

Range of $\sin^{-1}$ Domain of $\sin^{-1}$

Definition The inverse sine function, denoted by $\sin^{-1}$, is defined by $y = \sin^{-1} x$ if and only if $\sin y = x$, where $-1 \le x \le 1$ and $-\dfrac{\pi}{2} \le y \le \dfrac{\pi}{2}$.

Notation: Sometimes, $\sin^{-1} x$ is denoted by $Arc\sin x$. That is, $Arc\sin x = \sin^{-1} x$.

NOTE: The angles that are in the interval $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$ are the following angles in the *xy*-plane.



Thus, the angles that are used for the inverse sine function have their terminal side

1. In the **first quadrant** and are measured going **counterclockwise**.
2. In the **fourth quadrant** and are measured going **clockwise**.
3. On the positive *x*-axis. This is the angle of 0.
4. On the positive *y*-axis. This is the angle of $\dfrac{\pi}{2}$.
5. On the negative *y*-axis. This is the angle of $-\dfrac{\pi}{2}$.

Examples Find the exact value of the following.

1. $\sin^{-1} \dfrac{\sqrt{3}}{2}$

Since $\dfrac{\sqrt{3}}{2}$ is **positive**, then the angle answer is in the **I quadrant**. Which of the three angles of $\dfrac{\pi}{6}$, $\dfrac{\pi}{4}$, or $\dfrac{\pi}{3}$ would you be able to take the sine of and get $\dfrac{\sqrt{3}}{2}$? Or in terms of Unit Circle Trigonometry, which one of these three angles intersections the Unit Circle so that the *y*-coordinate of the point of intersection is $\dfrac{\sqrt{3}}{2}$? The answer is $\dfrac{\pi}{3}$.

There are two reasons that the answer is $\dfrac{\pi}{3}$:

1. $\sin \dfrac{\pi}{3} = \dfrac{\sqrt{3}}{2}$ and 2. $-\dfrac{\pi}{2} \le \dfrac{\pi}{3} \le \dfrac{\pi}{2}$

Answer: $\dfrac{\pi}{3}$

2. $Arc\sin \dfrac{\sqrt{2}}{2}$

Since $\dfrac{\sqrt{2}}{2}$ is **positive**, then the angle answer is in the **I quadrant**. Which of the three angles of $\dfrac{\pi}{6}$, $\dfrac{\pi}{4}$, or $\dfrac{\pi}{3}$ would you be able to take the sine of and get $\dfrac{\sqrt{2}}{2}$? Or in terms of Unit Circle Trigonometry, which one of these three angles intersections the Unit Circle so that the *y*-coordinate of the point of intersection is $\dfrac{\sqrt{2}}{2}$? The answer is $\dfrac{\pi}{4}$.

There are two reasons that the answer is $\dfrac{\pi}{4}$:

1. $\sin\dfrac{\pi}{4} = \dfrac{\sqrt{2}}{2}$ and 2. $-\dfrac{\pi}{2} \le \dfrac{\pi}{4} \le \dfrac{\pi}{2}$

Answer: $\dfrac{\pi}{4}$

3. $\sin^{-1}\left(-\dfrac{1}{2}\right)$

Since $-\dfrac{1}{2}$ is **negative**, then the angle answer is in the **IV quadrant**. Since the angle is not in the I quadrant, let's find the **reference angle** for it.

Let $\theta = \sin^{-1}\left(-\dfrac{1}{2}\right)$. Then $\theta' = \sin^{-1}\dfrac{1}{2}$

Which of the three angles of $\dfrac{\pi}{6}$, $\dfrac{\pi}{4}$, or $\dfrac{\pi}{3}$ would you be able to take the sine of and get $\dfrac{1}{2}$? Or in terms of Unit Circle Trigonometry, which one of these three angles intersections the Unit Circle so that the *y*-coordinate of the point of intersection is $\dfrac{1}{2}$? The answer is $\dfrac{\pi}{6}$.

Thus, $\theta' = \sin^{-1}\frac{1}{2} = \frac{\pi}{6}$. Now, go to the IV quadrant with this reference angle and measure the angle $\theta = \sin^{-1}\left(-\frac{1}{2}\right)$ going **clockwise**. Thus, $\sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$.

There are two reasons that the answer is $-\frac{\pi}{6}$:

1. $\sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$ and 2. $-\frac{\pi}{2} \le -\frac{\pi}{6} \le \frac{\pi}{2}$

Answer: $-\frac{\pi}{6}$

4. $Arc\sin\left(-\frac{\sqrt{3}}{2}\right)$

Since $-\frac{\sqrt{3}}{2}$ is **negative**, then the angle answer is in the **IV quadrant**. Since the angle is not in the I quadrant, let's find the **reference angle** for it.

Let $\theta = Arc\sin\left(-\frac{\sqrt{3}}{2}\right)$. Then $\theta' = Arc\sin\frac{\sqrt{3}}{2}$

Which of the three angles of $\frac{\pi}{6}$, $\frac{\pi}{4}$, or $\frac{\pi}{3}$ would you be able to take the sine of and get $\frac{\sqrt{3}}{2}$? Or in terms of Unit Circle Trigonometry, which one of these three angles intersections the Unit Circle so that the *y*-coordinate of the point of intersection is $\frac{\sqrt{3}}{2}$? The answer is $\frac{\pi}{3}$.

Thus, $\theta' = Arc\sin\frac{\sqrt{3}}{2} = \frac{\pi}{3}$. Now, go to the IV quadrant with this reference angle and measure the angle $\theta = Arc\sin\left(-\frac{\sqrt{3}}{2}\right)$ going **clockwise**. Thus, $Arc\sin\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$

There are two reasons that the answer is $-\frac{\pi}{3}$:

1. $\sin\left(-\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$ and 2. $-\frac{\pi}{2} \le -\frac{\pi}{3} \le \frac{\pi}{2}$

Answer: $-\frac{\pi}{3}$

5. $\sin^{-1} 0$

Since 0 is **not positive**, then the angle answer is **not** in the **I quadrant**. Since 0 is **not negative**, then the angle answer is **not** in the **IV quadrant**. Thus, the angle answer comes from one of the coordinate axis. It's either the negative *y*-axis, the positive *x*-axis, or the positive *y*-axis. Which of the three angles of $-\frac{\pi}{2}$, 0, or $\frac{\pi}{2}$ would you be able to take the sine of and get 0? Or in terms of Unit Circle Trigonometry, which one of these three angles intersections the Unit Circle so that the *y*-coordinate of the point of intersection is 0? The answer is 0.

There are two reasons that the answer is 0:

1. $\sin 0 = 0$ and 2. $-\frac{\pi}{2} \le 0 \le \frac{\pi}{2}$

Answer: 0

6. $Arc\sin 1$

Since 1 is the **maximum positive number**, then the angle answer does **not** come from the I quadrant. The angle answer comes from one of the coordinate axis. It's either the negative *y*-axis, the positive *x*-axis, or the positive *y*-axis. Which of the three angles of $-\frac{\pi}{2}$, 0, or $\frac{\pi}{2}$ would you be able to take the sine of and get 1? Or in terms of Unit Circle Trigonometry, which one of these three angles intersections the Unit Circle so that the *y*-coordinate of the point of intersection is 1? The answer is $\frac{\pi}{2}$.

There are two reasons that the answer is $\frac{\pi}{2}$:

1. $\sin\frac{\pi}{2} = 1$ and 2. $-\frac{\pi}{2} \le \frac{\pi}{2} \le \frac{\pi}{2}$

Answer: $\frac{\pi}{2}$

7. $\sin^{-1}(-1)$

Since -1 is the **minimum negative number**, then the angle answer does **not** come from the IV quadrant. The angle answer comes from one of the coordinate axis. It's either the negative *y*-axis, the positive *x*-axis, or the positive *y*-axis. Which of the three angles of $-\frac{\pi}{2}$, 0, or $\frac{\pi}{2}$ would you be able to take the sine of and get -1? Or in terms of Unit Circle Trigonometry, which one of these three angles intersections the Unit Circle so that the *y*-coordinate of the point of intersection is -1? The answer is $-\frac{\pi}{2}$.

There are two reasons that the answer is $-\dfrac{\pi}{2}$:

1. $\sin\left(-\dfrac{\pi}{2}\right) = -1$ and 2. $-\dfrac{\pi}{2} \le -\dfrac{\pi}{2} \le \dfrac{\pi}{2}$

Answer: $-\dfrac{\pi}{2}$

8. $Arc\sin\dfrac{1}{2}$

Since $\dfrac{1}{2}$ is **positive**, then the angle answer is in the **I quadrant**. Which of the three angles of $\dfrac{\pi}{6}$, $\dfrac{\pi}{4}$, or $\dfrac{\pi}{3}$ would you be able to take the sine of and get $\dfrac{1}{2}$? Or in terms of Unit Circle Trigonometry, which one of these three angles intersections the Unit Circle so that the *y*-coordinate of the point of intersection is $\dfrac{1}{2}$? The answer is $\dfrac{\pi}{6}$.

There are two reasons that the answer is $\dfrac{\pi}{6}$:

1. $\sin\dfrac{\pi}{6} = \dfrac{1}{2}$ and 2. $-\dfrac{\pi}{2} \le \dfrac{\pi}{6} \le \dfrac{\pi}{2}$

Answer: $\dfrac{\pi}{6}$

9. $\sin^{-1}\left(-\dfrac{\sqrt{2}}{2}\right)$

Since $-\frac{\sqrt{2}}{2}$ is **negative**, then the angle answer is in the **IV quadrant**. Since the angle is not in the I quadrant, let's find the **reference angle** for it.

Let $\theta = \sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$. Then $\theta' = \sin^{-1}\frac{\sqrt{2}}{2}$

Which of the three angles of $\frac{\pi}{6}$, $\frac{\pi}{4}$, or $\frac{\pi}{3}$ would you be able to take the sine of and get $\frac{\sqrt{2}}{2}$? Or in terms of Unit Circle Trigonometry, which one of these three angles intersections the Unit Circle so that the *y*-coordinate of the point of intersection is $\frac{\sqrt{2}}{2}$? The answer is $\frac{\pi}{4}$.

Thus, $\theta' = \sin^{-1}\frac{\sqrt{2}}{2} = \frac{\pi}{4}$. Now, go to the IV quadrant with this reference angle and measure the angle $\theta = \sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$ going **clockwise**. Thus,

$\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$

There are two reasons that the answer is $-\frac{\pi}{4}$:

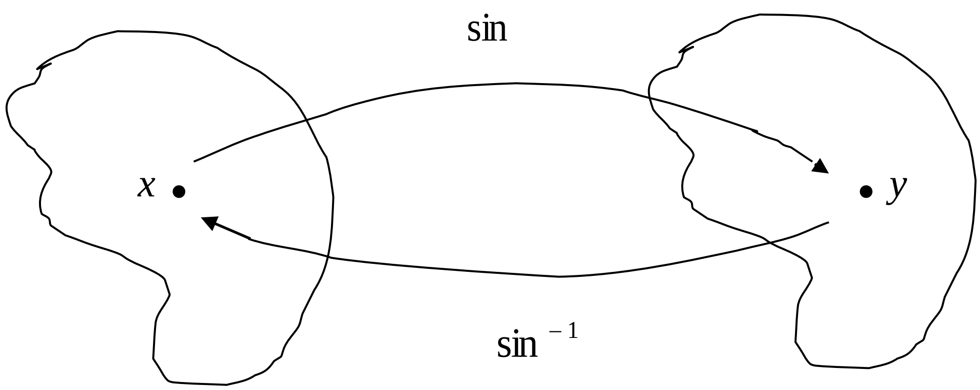
1. $\sin\left(-\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$ and 2. $-\frac{\pi}{2} \leq -\frac{\pi}{4} \leq \frac{\pi}{2}$

Answer: $-\frac{\pi}{4}$

Back to Topics List

3. RELATIONSHIPS BETWEEN THE SINE AND INVERSE SINE FUNCTIONS

From Section 1 above, we have the following relationships:



Restricted domain of $\sin = \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$ $\qquad$ Range of $\sin = [-1, 1]$

$\qquad\qquad ||$ $\qquad\qquad\qquad\qquad\qquad\qquad ||$

Range of $\sin^{-1}$ $\qquad\qquad\qquad\qquad$ Domain of $\sin^{-1}$

We also have the following two relationships between the sine function and its inverse sine function:

1. $\sin^{-1}(\sin x) = x$ for all x in the **restricted domain of** $\sin = \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$

2. $\sin(\sin^{-1} y) = y$ for all y in the **domain of** $\sin^{-1} = [-1, 1]$

Examples Find the exact value of the following.

1. $\sin^{-1}\left(\sin \dfrac{\pi}{3}\right)$

Note that $\frac{\pi}{3}$ is an angle such that $-\frac{\pi}{2} \leq \frac{\pi}{3} \leq \frac{\pi}{2}$. Thus, $\sin^{-1}\left(\sin\frac{\pi}{3}\right) = \frac{\pi}{3}$ by Relationship 1 above.

Another way to find the answer is by first finding the value of $\sin\frac{\pi}{3}$ and then finding the inverse sine of this value: Since $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$, then $\sin^{-1}\left(\sin\frac{\pi}{3}\right) = \sin^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{3}$.

Answer: $\frac{\pi}{3}$

2. $\sin\left[\sin^{-1}\left(-\frac{1}{2}\right)\right]$

Note that $\sin^{-1}\left(-\frac{1}{2}\right)$ is a defined angle if $-\frac{1}{2}$ is a number such that $-1 \leq -\frac{1}{2} \leq 1$. Since $-1 \leq -\frac{1}{2} \leq 1$, then $\sin^{-1}\left(-\frac{1}{2}\right)$ is a defined angle. Thus, $\sin\left[\sin^{-1}\left(-\frac{1}{2}\right)\right] = -\frac{1}{2}$ by Relationship 2 above.

Another way to find the answer is by first finding the angle $\sin^{-1}\left(-\frac{1}{2}\right)$ and then finding the sine of this angle: Since $\sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$, then $\sin\left[\sin^{-1}\left(-\frac{1}{2}\right)\right] = \sin\left(-\frac{\pi}{6}\right) = -\sin\frac{\pi}{6} = -\frac{1}{2}$.

Answer: $-\dfrac{1}{2}$

3. $Arc\sin\left(\sin\dfrac{2\pi}{5}\right)$

Since $\dfrac{2\pi}{5}$ is an angle such that $-\dfrac{\pi}{2} \le \dfrac{2\pi}{5} \le \dfrac{\pi}{2}$, then $Arc\sin\left(\sin\dfrac{2\pi}{5}\right) = \dfrac{2\pi}{5}$ by Relationship 1 above.

Answer: $\dfrac{2\pi}{5}$

4. $\sin\left(Arc\sin\dfrac{7}{9}\right)$

Note that $Arc\sin\dfrac{7}{9}$ is a defined angle if $\dfrac{7}{9}$ is a number such that $-1 \le \dfrac{7}{9} \le 1$. Since $-1 \le \dfrac{7}{9} \le 1$, then $Arc\sin\dfrac{7}{9}$ is a defined angle. Thus, $\sin\left(Arc\sin\dfrac{7}{9}\right) = \dfrac{7}{9}$ by Relationship 2 above.

Answer: $\dfrac{7}{9}$

5. $\sin^{-1}[\sin(-1.5)]$

Since -1.5 is an angle such that $-\dfrac{\pi}{2} \le -1.5 \le \dfrac{\pi}{2}$, then $\sin^{-1}[\sin(-1.5)] = -1.5$ by Relationship 1 above.

Answer: -1.5

6. $\sin\left[\sin^{-1}\left(-\frac{\pi}{2}\right)\right]$

Note that $\sin^{-1}\left(-\frac{\pi}{2}\right)$ is a defined angle if $-\frac{\pi}{2}$ is a number such that $-1 \leq -\frac{\pi}{2} \leq 1$. Since $-\frac{\pi}{2} \approx -1.57$, then $-\frac{\pi}{2} < -1$. Thus, $\sin^{-1}\left(-\frac{\pi}{2}\right)$ is not a defined angle. Thus, $\sin\left[\sin^{-1}\left(-\frac{\pi}{2}\right)\right]$ is undefined.

Answer: undefined

7. $\sin\left[\sin^{-1}\frac{\pi}{4}\right]$

Note that $\sin^{-1}\frac{\pi}{4}$ is a defined angle if $\frac{\pi}{4}$ is a number such that $-1 \leq \frac{\pi}{4} \leq 1$. Since $\frac{\pi}{4} \approx 0.79$, then $-1 \leq \frac{\pi}{4} \leq 1$. Thus, $\sin^{-1}\frac{\pi}{4}$ is a defined angle and $\sin\left[\sin^{-1}\frac{\pi}{4}\right] = \frac{\pi}{4}$ by Relationship 2 above.

Answer: $\frac{\pi}{4}$

8. $\sin^{-1}\left(\sin\frac{5\pi}{6}\right)$

Note that $\frac{5\pi}{6}$ is an angle such that $\frac{5\pi}{6} > \frac{\pi}{2}$. Thus, Relationship 1 above does not apply. There are two ways to find the answer.

The first way is to find the value of $\sin \frac{5\pi}{6}$: Since $\sin \frac{5\pi}{6} = \sin \frac{\pi}{6} = \frac{1}{2}$, then $\sin^{-1}\left(\sin \frac{5\pi}{6}\right) = \sin^{-1}\frac{1}{2} = \frac{\pi}{6}$.

The second way is to use reference angles: Since $\sin \frac{5\pi}{6} = \sin \frac{\pi}{6}$, then $\sin^{-1}\left(\sin \frac{5\pi}{6}\right) = \sin^{-1}\left(\sin \frac{\pi}{6}\right) = \frac{\pi}{6}$ by Relationship 1 above since $-\frac{\pi}{2} \leq \frac{\pi}{6} \leq \frac{\pi}{2}$.

Answer: $\frac{\pi}{6}$

9. $Arc\sin\left[\sin\left(-\frac{3\pi}{4}\right)\right]$

Note that $-\frac{3\pi}{4}$ is an angle such that $-\frac{3\pi}{4} < -\frac{\pi}{2}$. Thus, Relationship 1 above does not apply. There are two ways to find the answer.

The first way is to find the value of $\sin\left(-\frac{3\pi}{4}\right)$: Since $\sin\left(-\frac{3\pi}{4}\right) = -\sin\frac{\pi}{4} = -\frac{\sqrt{2}}{2}$, then $Arc\sin\left[\sin\left(-\frac{3\pi}{4}\right)\right] = Arc\sin\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$.

The second way is to use reference angles: Since $\sin\left(-\frac{3\pi}{4}\right) = -\sin\frac{\pi}{4} = \sin\left(-\frac{\pi}{4}\right)$ then $Arc\sin\left[\sin\left(-\frac{3\pi}{4}\right)\right] = Arc\sin\left[\sin\left(-\frac{\pi}{4}\right)\right] = -\frac{\pi}{4}$ by Relationship 1 above since $-\frac{\pi}{2} \le -\frac{\pi}{4} \le \frac{\pi}{2}$.

Answer: $-\frac{\pi}{4}$

10. $\sin^{-1}\left(\sin\frac{15\pi}{11}\right)$

Note that $\frac{15\pi}{11}$ is an angle such that $\frac{15\pi}{11} > \frac{\pi}{2}$. Thus, Relationship 1 above does not apply. Since $\frac{15\pi}{11}$ is not one of our special angles, then we will not be able to find the exact value of $\sin\frac{15\pi}{11}$. Thus, we will need to use reference angles to find the answer.

The angle $\frac{15\pi}{11}$ is in the III quadrant and the reference angle for this angle is $\frac{4\pi}{11}$. Since $\sin\frac{15\pi}{11} = -\sin\frac{4\pi}{11} = \sin\left(-\frac{4\pi}{11}\right)$, then $\sin^{-1}\left(\sin\frac{15\pi}{11}\right) = \sin^{-1}\left[\sin\left(-\frac{4\pi}{11}\right)\right] = -\frac{4\pi}{11}$ by Relationship 1above since $-\frac{\pi}{2} \le -\frac{4\pi}{11} \le \frac{\pi}{2}$.

Answer: $-\frac{4\pi}{11}$

11. $Arc\sin\left(\sin\dfrac{5\pi}{7}\right)$

Note that $\dfrac{5\pi}{7}$ is an angle such that $\dfrac{5\pi}{7} > \dfrac{\pi}{2}$. Thus, Relationship 1 above does not apply. Since $\dfrac{5\pi}{7}$ is not one of our special angles, then we will not be able to find the exact value of $\sin\dfrac{5\pi}{7}$. Thus, we will need to use reference angles to find the answer.

The angle $\dfrac{5\pi}{7}$ is in the II quadrant and the reference angle for this angle is $\dfrac{2\pi}{7}$. Since $\sin\dfrac{5\pi}{7} = \sin\dfrac{2\pi}{7}$, then $Arc\sin\left(\sin\dfrac{5\pi}{7}\right) =$
$Arc\sin\left(\sin\dfrac{2\pi}{7}\right) = \dfrac{2\pi}{7}$ by Relationship 1above since $-\dfrac{\pi}{2} \le \dfrac{2\pi}{7} \le \dfrac{\pi}{2}$.

Answer: $\dfrac{2\pi}{7}$

12. $\sin^{-1}\left[\sin\left(-\dfrac{17\pi}{10}\right)\right]$

Note that $-\dfrac{17\pi}{10}$ is an angle such that $-\dfrac{17\pi}{10} < -\dfrac{\pi}{2}$. Thus, Relationship 1 above does not apply. Since $-\dfrac{17\pi}{10}$ is not one of our special angles, then we will not be able to find the exact value of $\sin\left(-\dfrac{17\pi}{10}\right)$. Since the angle $-\dfrac{17\pi}{10}$ is in the I quadrant, there are two ways to find the answer.

The first way is to use reference angles. The reference angle for the angle $-\dfrac{17\pi}{10}$ is $\dfrac{3\pi}{10}$. Since $\sin\left(-\dfrac{17\pi}{10}\right) = \sin\dfrac{3\pi}{10}$, then $\sin^{-1}\left[\sin\left(-\dfrac{17\pi}{10}\right)\right] = \sin^{-1}\left(\sin\dfrac{3\pi}{10}\right) = \dfrac{3\pi}{10}$ by Relationship 1 above since $-\dfrac{\pi}{2} \le \dfrac{3\pi}{10} \le \dfrac{\pi}{2}$.

The second way is to use a coterminal angle. Since the angle $-\dfrac{17\pi}{10}$ is in the I quadrant, then find the positive angle between 0 and $\dfrac{\pi}{2}$ that is coterminal with $-\dfrac{17\pi}{10}$. This angle is obtained by adding 2π to $-\dfrac{17\pi}{10}$. Thus, $-\dfrac{17\pi}{10} + 2\pi = -\dfrac{17\pi}{10} + \dfrac{20\pi}{10} = \dfrac{3\pi}{10}$. Since the angle $-\dfrac{17\pi}{10}$ is coterminal with the angle $\dfrac{3\pi}{10}$, then $\sin\left(-\dfrac{17\pi}{10}\right) = \sin\dfrac{3\pi}{10}$. Thus, $\sin^{-1}\left[\sin\left(-\dfrac{17\pi}{10}\right)\right] = \sin^{-1}\left(\sin\dfrac{3\pi}{10}\right) = \dfrac{3\pi}{10}$ by Relationship 1 above since $-\dfrac{\pi}{2} \le \dfrac{3\pi}{10} \le \dfrac{\pi}{2}$.

NOTE: This second method will work for any negative angle, between -2π and 0, whose terminal side is in the I quadrant. Let θ be such a negative angle in the I quadrant. Since the angle θ is in the I quadrant, then $-2\pi < \theta < -\dfrac{3\pi}{2}$. Adding 2π to both sides of this compound inequality, we obtain that $-2\pi + 2\pi < \theta + 2\pi < -\dfrac{3\pi}{2} + 2\pi$. Thus, $0 < \theta + 2\pi < \dfrac{\pi}{2}$. Since the angles θ and $\theta + 2\pi$ are coterminal, then $\sin\theta = \sin(\theta + 2\pi)$. Thus, $Arc\sin(\sin\theta) = Arc\sin[\sin(\theta + 2\pi)] = \theta + 2\pi$ by Relationship 1 above since $0 < \theta + 2\pi < \dfrac{\pi}{2}$.

As we will see in our last example, this second method will also work for a positive angle, between 0 and 2π, whose terminal side is in the IV quadrant. However, this second method will not work for any angle, between -2π and 2π, whose terminal side is in either the II or III quadrant.

Answer: $\dfrac{3\pi}{10}$

13. $Arc\sin\left(\sin\dfrac{31\pi}{18}\right)$

Note that $\dfrac{31\pi}{18}$ is an angle such that $\dfrac{31\pi}{18} > \dfrac{\pi}{2}$. Thus, Relationship 1 above does not apply. Since $\dfrac{31\pi}{18}$ is not one of our special angles, then we will not be able to find the exact value of $\sin\dfrac{31\pi}{18}$. Since the angle $\dfrac{31\pi}{18}$ is in the IV quadrant, there are two ways to find the answer.

The first way is to use reference angles. The reference angle for the angle $\dfrac{31\pi}{18}$ is $\dfrac{5\pi}{18}$. Since $\sin\dfrac{31\pi}{18} = -\sin\dfrac{5\pi}{18} = \sin\left(-\dfrac{5\pi}{18}\right)$, then $Arc\sin\left(\sin\dfrac{31\pi}{18}\right) = Arc\sin\left[\sin\left(-\dfrac{5\pi}{18}\right)\right] = -\dfrac{5\pi}{18}$ by Relationship 1 above since $-\dfrac{\pi}{2} \leq -\dfrac{5\pi}{18} \leq \dfrac{\pi}{2}$.

The second way is to use a coterminal angle. Since the angle $\dfrac{31\pi}{18}$ is in the IV quadrant, then find the negative angle between $-\dfrac{\pi}{2}$ and 0 that is coterminal with $\dfrac{31\pi}{18}$. This angle is obtained by subtracting 2π from $\dfrac{31\pi}{18}$.

Thus, $\frac{31\pi}{18} - 2\pi = \frac{31\pi}{18} - \frac{36\pi}{18} = -\frac{5\pi}{18}$. Since the angle $\frac{31\pi}{18}$ is coterminal with the angle $-\frac{5\pi}{18}$, then $\sin\frac{31\pi}{18} = \sin\left(-\frac{5\pi}{18}\right)$. Thus, $Arc\sin\left(\sin\frac{31\pi}{18}\right) = Arc\sin\left[\sin\left(-\frac{5\pi}{18}\right)\right] = -\frac{5\pi}{18}$ by Relationship 1 above since $-\frac{\pi}{2} \leq -\frac{5\pi}{18} \leq \frac{\pi}{2}$.

NOTE: This second method will work for any positive angle, between 0 and 2π, whose terminal side is in the IV quadrant. Let θ be such a positive angle in the IV quadrant. Since the angle θ is in the IV quadrant, then $\frac{3\pi}{2} < \theta < 2\pi$. Subtracting 2π from both sides of this compound inequality, we obtain that $\frac{3\pi}{2} - 2\pi < \theta - 2\pi < 2\pi - 2\pi$. Thus, $-\frac{\pi}{2} < \theta - 2\pi < 0$. Since the angles θ and $\theta - 2\pi$ are coterminal, then $\sin\theta = \sin(\theta - 2\pi)$. Thus, $Arc\sin(\sin\theta) = Arc\sin[\sin(\theta - 2\pi)] = \theta - 2\pi$ by Relationship 1 above since $-\frac{\pi}{2} < \theta - 2\pi < 0$.

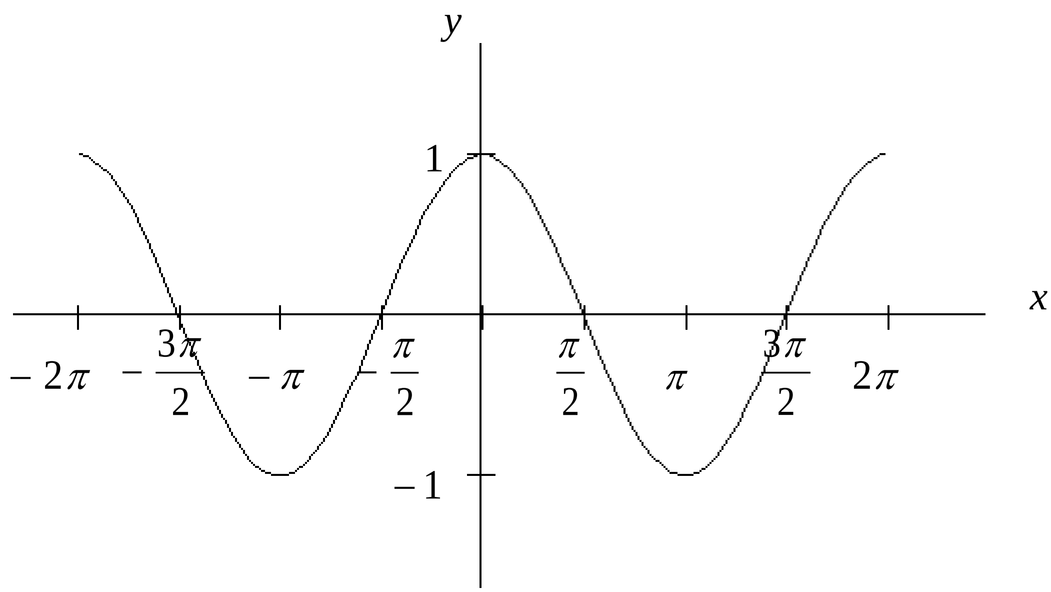
Again, this second method will not work for any angle, between -2π and 2π, whose terminal side is in either the II or III quadrant.

Answer: $-\frac{5\pi}{18}$

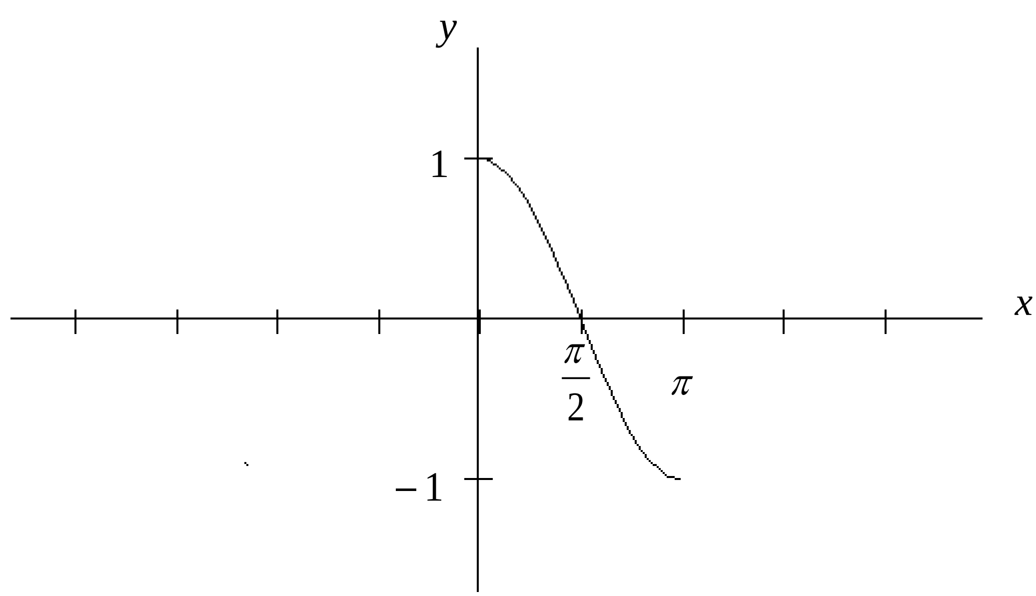
Back to Topics List

4. THE INVERSE COSINE FUNCTION

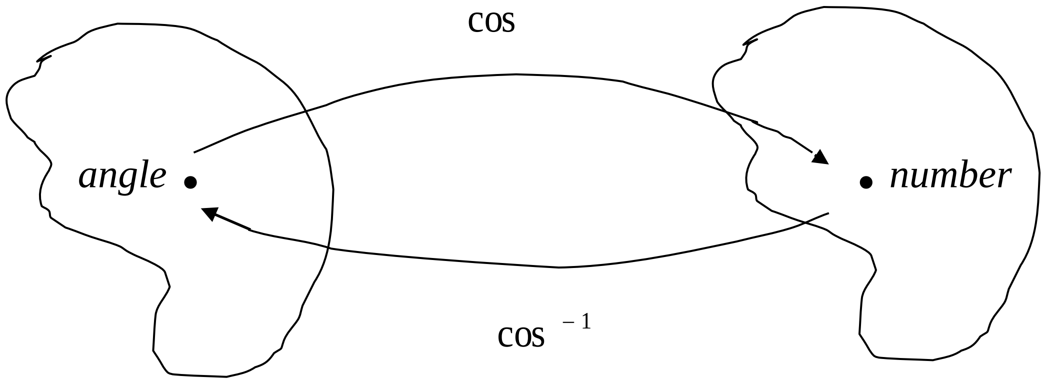
Consider the following two cycles of the graph of $y = \cos x$:



The range of $y = \cos x$ is the interval $[-1, 1]$. Thus, the domain of the inverse cosine function is $[-1, 1]$. The domain of $y = \cos x$ is the set of all real numbers. However, by the horizontal line test, the function $y = \cos x$ is not one-to-one. Thus, we will need to put a restriction of the domain. We will restrict the domain of the function $y = \cos x$ to the interval of angles $[0, \pi]$. The graph of $y = \cos x$ on the restricted domain looks like the following:



Since the restricted domain of $y = \cos x$ is the interval of angles $[0, \pi]$, then range of the inverse cosine function is the interval of angles $[0, \pi]$.



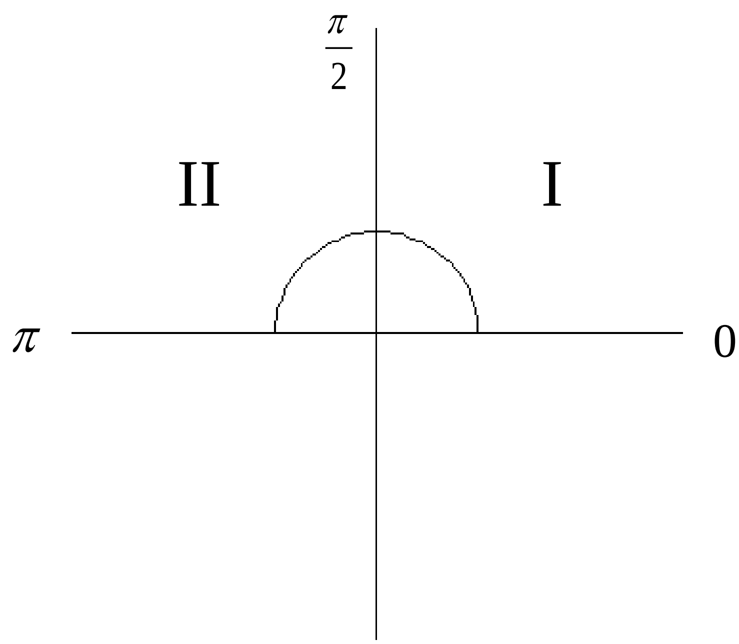
Restricted domain of $\cos = [0, \pi]$ ‖ Range of $\cos^{-1}$

Range of $\cos = [-1, 1]$ ‖ Domain of $\cos^{-1}$

Definition The inverse cosine function, denoted by $\cos^{-1}$, is defined by $y = \cos^{-1} x$ if and only if $\cos y = x$, where $-1 \leq x \leq 1$ and $0 \leq y \leq \pi$.

Notation: Sometimes, $\cos^{-1}$ is denoted by $Arc\cos x$. That is, $Arc\cos x = \cos^{-1} x$.

NOTE: The angles that are in the interval $[0, \pi]$ are the following angles in the *xy*-plane.



Thus, the angles that are used for the inverse cosine function have their terminal side

1. In the **first quadrant** and are measured going **counterclockwise**.
2. In the **second quadrant** and are measured going **counterclockwise**.
3. On the positive x-axis. This is the angle of 0.
4. On the positive y-axis. This is the angle of $\frac{\pi}{2}$.
5. On the negative x-axis. This is the angle of π.

<u>Examples</u> Find the exact value of the following.

1. $\cos^{-1} \frac{\sqrt{3}}{2}$

Since $\frac{\sqrt{3}}{2}$ is **positive**, then the angle answer is in the **I quadrant**. Which of the three angles of $\frac{\pi}{6}$, $\frac{\pi}{4}$, or $\frac{\pi}{3}$ would you be able to take the cosine of and get $\frac{\sqrt{3}}{2}$? Or in terms of Unit Circle Trigonometry, which one of these three angles intersections the Unit Circle so that the x-coordinate of the point of intersection is $\frac{\sqrt{3}}{2}$? The answer is $\frac{\pi}{6}$.

There are two reasons that the answer is $\frac{\pi}{6}$:

1. $\cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$ and 2. $0 \le \frac{\pi}{6} \le \pi$

Answer: $\frac{\pi}{6}$

2. $Arc\cos\dfrac{\sqrt{2}}{2}$

Since $\dfrac{\sqrt{2}}{2}$ is **positive**, then the angle answer is in the **I quadrant**. Which of the three angles of $\dfrac{\pi}{6}$, $\dfrac{\pi}{4}$, or $\dfrac{\pi}{3}$ would you be able to take the cosine of and get $\dfrac{\sqrt{2}}{2}$? Or in terms of Unit Circle Trigonometry, which one of these three angles intersections the Unit Circle so that the *x*-coordinate of the point of intersection is $\dfrac{\sqrt{2}}{2}$? The answer is $\dfrac{\pi}{4}$.

There are two reasons that the answer is $\dfrac{\pi}{4}$:

1. $\cos\dfrac{\pi}{4} = \dfrac{\sqrt{2}}{2}$ and 2. $0 \le \dfrac{\pi}{4} \le \pi$

Answer: $\dfrac{\pi}{4}$

3. $\cos^{-1}\left(-\dfrac{1}{2}\right)$

Since $-\dfrac{1}{2}$ is **negative**, then the angle answer is in the **II quadrant**. Since the angle is not in the I quadrant, let's find the **reference angle** for it.

$$\text{Let } \theta = \cos^{-1}\left(-\frac{1}{2}\right). \text{ Then } \theta' = \cos^{-1}\frac{1}{2}$$

Which of the three angles of $\dfrac{\pi}{6}$, $\dfrac{\pi}{4}$, or $\dfrac{\pi}{3}$ would you be able to take the cosine of and get $\frac{1}{2}$? Or in terms of Unit Circle Trigonometry, which one of these three angles intersections the Unit Circle so that the *x*-coordinate of the point of intersection is $\frac{1}{2}$? The answer is $\frac{\pi}{3}$.

Thus, $\theta' = \cos^{-1}\frac{1}{2} = \frac{\pi}{3}$. Now, go to the II quadrant with this reference angle and measure the angle $\theta = \cos^{-1}\left(-\frac{1}{2}\right)$ going **counterclockwise**. Thus, $\cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$.

There are two reasons that the answer is $\frac{2\pi}{3}$:

1. $\cos\frac{2\pi}{3} = -\frac{1}{2}$ and 2. $0 \le \frac{2\pi}{3} \le \pi$

Answer: $\frac{2\pi}{3}$

4. $Arc\cos\left(-\frac{\sqrt{3}}{2}\right)$

Since $-\frac{\sqrt{3}}{2}$ is **negative**, then the angle answer is in the **II quadrant**. Since the angle is not in the I quadrant, let's find the **reference angle** for it.

Let $\theta = Arc\cos\left(-\frac{\sqrt{3}}{2}\right)$. Then $\theta' = Arc\cos\frac{\sqrt{3}}{2}$

Which of the three angles of $\frac{\pi}{6}$, $\frac{\pi}{4}$, or $\frac{\pi}{3}$ would you be able to take the cosine of and get $\frac{\sqrt{3}}{2}$? Or in terms of Unit Circle Trigonometry, which one of these three angles intersections the Unit Circle so that the *x*-coordinate of the point of intersection is $\frac{\sqrt{3}}{2}$? The answer is $\frac{\pi}{6}$.

Thus, $\theta' = Arc\cos\frac{\sqrt{3}}{2} = \frac{\pi}{6}$. Now, go to the II quadrant with this reference angle and measure the angle $\theta = Arc\cos\left(-\frac{\sqrt{3}}{2}\right)$ going **counterclockwise**.

Thus, $Arc\cos\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$

There are two reasons that the answer is $\frac{5\pi}{6}$:

1. $\cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2}$ and 2. $0 \le \frac{5\pi}{6} \le \pi$

Answer: $\frac{5\pi}{6}$

5. $\cos^{-1} 0$

Since 0 is **not positive**, then the angle answer is **not** in the **I quadrant**. Since 0 is **not negative**, then the angle answer is **not** in the **II quadrant**. Thus, the angle answer comes from one of the coordinate axis. It's either the positive *x*-axis, the positive *y*-axis, or the negative *x*-axis. Which of the three angles of 0, $\frac{\pi}{2}$, or π would you be able to take the cosine of and get 0? Or in terms of Unit Circle Trigonometry, which one of these three angles intersections the Unit Circle so that the x-coordinate of the point of intersection is 0? The answer is $\frac{\pi}{2}$.

There are two reasons that the answer is $\frac{\pi}{2}$:

1. $\cos \frac{\pi}{2} = 0$ and 2. $0 \leq \frac{\pi}{2} \leq \pi$

Answer: $\frac{\pi}{2}$

6. $Arc\cos 1$

Since 1 is the **maximum positive number**, then the angle answer does **not** come from the I quadrant. The angle answer comes from one of the coordinate axis. It's either the positive x-axis, the positive y-axis, or the negative x-axis. Which of the three angles of 0, $\frac{\pi}{2}$, or π would you be able to take the cosine of and get 1? Or in terms of Unit Circle Trigonometry, which one of these three angles intersections the Unit Circle so that the x-coordinate of the point of intersection is 1? The answer is 0.

There are two reasons that the answer is 0:

1. $\cos 0 = 1$ and 2. $0 \leq 0 \leq \pi$

Answer: 0

7. $\cos^{-1}(-1)$

Since -1 is the **minimum negative number**, then the angle answer does **not** come from the II quadrant. The angle answer comes from one of the coordinate axis. It's either the positive x-axis, the positive y-axis, or the negative x-axis. Which of the three angles of 0, $\dfrac{\pi}{2}$, or π would you be able to take the cosine of and get -1? Or in terms of Unit Circle Trigonometry, which one of these three angles intersections the Unit Circle so that the x-coordinate of the point of intersection is -1? The answer is π.

There are two reasons that the answer is π:

1. $\cos \pi = -1$ and 2. $0 \le \pi \le \pi$

Answer: π

8. $Arc\cos \dfrac{1}{2}$

Since $\dfrac{1}{2}$ is **positive**, then the angle answer is in the **I quadrant**. Which of the three angles of $\dfrac{\pi}{6}$, $\dfrac{\pi}{4}$, or $\dfrac{\pi}{3}$ would you be able to take the cosine of and get $\dfrac{1}{2}$? Or in terms of Unit Circle Trigonometry, which one of these three angles intersections the Unit Circle so that the x-coordinate of the point of intersection is $\dfrac{1}{2}$? The answer is $\dfrac{\pi}{3}$.

There are two reasons that the answer is $\dfrac{\pi}{3}$:

1. $\cos \dfrac{\pi}{3} = \dfrac{1}{2}$ and 2. $0 \le \dfrac{\pi}{3} \le \pi$

Answer: $\dfrac{\pi}{3}$

9. $\cos^{-1}\left(-\dfrac{\sqrt{2}}{2}\right)$

Since $-\dfrac{\sqrt{2}}{2}$ is **negative**, then the angle answer is in the **II quadrant**. Since the angle is not in the I quadrant, let's find the **reference angle** for it.

$$\text{Let } \theta = \cos^{-1}\left(-\frac{\sqrt{2}}{2}\right). \text{ Then } \theta' = \cos^{-1}\frac{\sqrt{2}}{2}$$

Which of the three angles of $\dfrac{\pi}{6}$, $\dfrac{\pi}{4}$, or $\dfrac{\pi}{3}$ would you be able to take the cosine of and get $\dfrac{\sqrt{2}}{2}$? Or in terms of Unit Circle Trigonometry, which one of these three angles intersections the Unit Circle so that the *x*-coordinate of the point of intersection is $\dfrac{\sqrt{2}}{2}$? The answer is $\dfrac{\pi}{4}$.

Thus, $\theta' = \cos^{-1}\dfrac{\sqrt{2}}{2} = \dfrac{\pi}{4}$. Now, go to the II quadrant with this reference angle and measure the angle $\theta = \cos^{-1}\left(-\dfrac{\sqrt{2}}{2}\right)$ going **counterclockwise**.

Thus, $\cos^{-1}\left(-\dfrac{\sqrt{2}}{2}\right) = \dfrac{3\pi}{4}$

There are two reasons that the answer is $\dfrac{3\pi}{4}$:

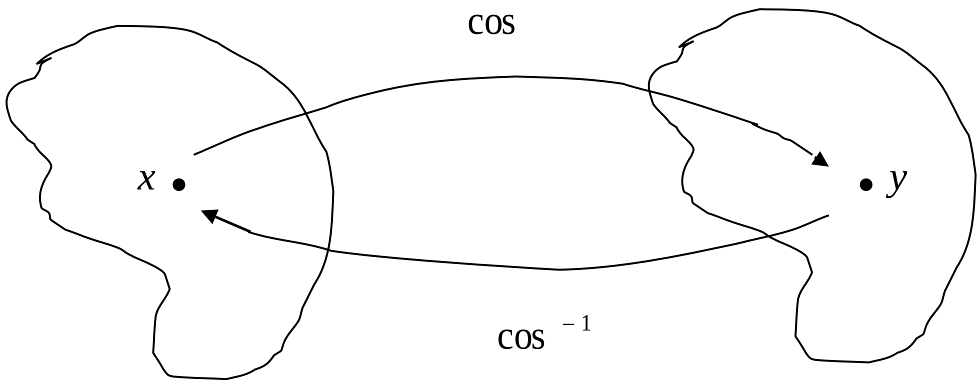
1. $\cos\dfrac{3\pi}{4} = -\dfrac{\sqrt{2}}{2}$ and 2. $0 \le \dfrac{3\pi}{4} \le \pi$

Answer: $\dfrac{3\pi}{4}$

Back to Topics List

5. RELATIONSHIPS BETWEEN THE COSINE AND INVERSE COSINE FUNCTIONS

From Section 1 above, we have the following relationships:



Restricted domain of $\cos = [0, \pi]$ Range of $\cos = [-1, 1]$

$\|$ $\|$

Range of $\cos^{-1}$ Domain of $\cos^{-1}$

We also have the following two relationships between the cosine function and its inverse cosine function:

1. $\cos^{-1}(\cos x) = x$ for all x in the **restricted domain of** $\cos = [0, \pi]$

2. $\cos(\cos^{-1} y) = y$ for all y in the **domain of** $\cos^{-1} = [-1, 1]$

Examples Find the exact value of the following.

1. $\cos^{-1}\left(\cos\frac{\pi}{4}\right)$

Note that $\frac{\pi}{4}$ is an angle such that $0 \le \frac{\pi}{4} \le \pi$. Thus, $\cos^{-1}\left(\cos\frac{\pi}{4}\right) = \frac{\pi}{4}$ by Relationship 1 above.

Another way to find the answer is by first finding the value of $\cos\frac{\pi}{4}$ and then finding the inverse cosine of this value: Since $\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$, then $\cos^{-1}\left(\cos\frac{\pi}{4}\right) = \cos^{-1}\frac{\sqrt{2}}{2} = \frac{\pi}{4}$.

Answer: $\frac{\pi}{4}$

2. $\cos\left[\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right]$

Note that $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ is a defined angle if $-\frac{\sqrt{3}}{2}$ is a number such that $-1 \le -\frac{\sqrt{3}}{2} \le 1$. Since $-1 \le -\frac{\sqrt{3}}{2} \le 1$, then $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ is a defined angle and $\cos\left[\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right] = -\frac{\sqrt{3}}{2}$ by Relationship 2 above.

Another way to find the answer is by first finding the angle $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ and then finding the cosine of this angle: Since $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$, then $\cos\left[\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right] = \cos\left(\frac{5\pi}{6}\right) = -\cos\frac{\pi}{6} = -\frac{\sqrt{3}}{2}$.

Answer: $-\frac{\sqrt{3}}{2}$

3. $Arc\cos\left(\cos\frac{5\pi}{7}\right)$

Since $\frac{5\pi}{7}$ is an angle such that $0 \le \frac{5\pi}{7} \le \pi$, then $Arc\cos\left(\cos\frac{5\pi}{7}\right) = \frac{5\pi}{7}$ by Relationship 1 above.

Answer: $\frac{5\pi}{7}$

4. $\cos\left[Arc\cos\left(-\frac{3}{5}\right)\right]$

Note that $Arc\cos\left(-\frac{3}{5}\right)$ is a defined angle if $-\frac{3}{5}$ is a number such that $-1 \le -\frac{3}{5} \le 1$. Since $-1 \le -\frac{3}{5} \le 1$, then $Arc\cos\left(-\frac{3}{5}\right)$ is a defined angle and $\cos\left[Arc\cos\left(-\frac{3}{5}\right)\right] = -\frac{3}{5}$ by Relationship 2 above.

Answer: $-\dfrac{3}{5}$

5. $\cos^{-1}(\cos 3)$

Since 3 is an angle such that $0 \le 3 \le \pi$, then, $\cos^{-1}(\cos 3) = 3$ by Relationship 1 above.

Answer: 3

6. $\cos\left(\cos^{-1}\dfrac{2\pi}{3}\right)$

Note that $\cos^{-1}\dfrac{2\pi}{3}$ is a defined angle if $\dfrac{2\pi}{3}$ is a number such that $-1 \le \dfrac{2\pi}{3} \le 1$. Since $\dfrac{2\pi}{3} \approx 2.09$, then $\dfrac{2\pi}{3} > 1$. Thus, $\cos^{-1}\dfrac{2\pi}{3}$ is not a defined angle. Thus, $\cos\left(\cos^{-1}\dfrac{2\pi}{3}\right)$ is undefined.

Answer: undefined

7. $\cos\left[\cos^{-1}\left(-\dfrac{\pi}{6}\right)\right]$

Note that $\cos^{-1}\left(-\dfrac{\pi}{6}\right)$ is a defined angle if $-\dfrac{\pi}{6}$ is a number such that $-1 \le -\dfrac{\pi}{6} \le 1$. Since $-\dfrac{\pi}{6} \approx 0.52$, then $-1 \le -\dfrac{\pi}{6} \le 1$. Thus, $\cos^{-1}\left(-\dfrac{\pi}{6}\right)$ is a defined angle and $\cos\left[\cos^{-1}\left(-\dfrac{\pi}{6}\right)\right] = -\dfrac{\pi}{6}$ by Relationship 2 above.

Answer: $-\dfrac{\pi}{6}$

8. $\cos^{-1}\left(\cos\dfrac{4\pi}{3}\right)$

Note that $\dfrac{4\pi}{3}$ is an angle such that $\dfrac{4\pi}{3} > \pi$. Thus, Relationship 1 above does not apply. There are two ways to find the answer.

The first way is to find the value of $\cos\dfrac{4\pi}{3}$: Since $\cos\dfrac{4\pi}{3} = -\cos\dfrac{\pi}{3} = -\dfrac{1}{2}$, then $\cos^{-1}\left(\cos\dfrac{4\pi}{3}\right) = \cos^{-1}\left(-\dfrac{1}{2}\right) = \dfrac{2\pi}{3}$.

The second way is to use reference angles: Since $\cos\dfrac{4\pi}{3} = -\cos\dfrac{\pi}{3} = \cos\dfrac{2\pi}{3}$, then $\cos^{-1}\left(\cos\dfrac{4\pi}{3}\right) = \cos^{-1}\left(\cos\dfrac{2\pi}{3}\right) = \dfrac{2\pi}{3}$ by Relationship 1 above since $0 \le \dfrac{2\pi}{3} \le \pi$.

Answer: $\dfrac{2\pi}{3}$

9. $Arc\cos\left(\cos\dfrac{11\pi}{6}\right)$

Note that $\dfrac{11\pi}{6}$ is an angle such that $\dfrac{11\pi}{6} > \pi$. Thus, Relationship 1 above does not apply. There are two ways to find the answer.

The first way is to find the value of $\cos\frac{11\pi}{6}$: Since $\cos\frac{11\pi}{6} = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$, then $Arc\cos\left(\cos\frac{11\pi}{6}\right) = Arc\cos\frac{\sqrt{3}}{2} = \frac{\pi}{6}$.

The second way is to use reference angles: Since $\cos\frac{11\pi}{6} = \cos\frac{\pi}{6}$, then $Arc\cos\left(\cos\frac{11\pi}{6}\right) = Arc\cos\left(\cos\frac{\pi}{6}\right) = \frac{\pi}{6}$ by Relationship 1 above since $0 \leq \frac{\pi}{6} \leq \pi$.

Answer: $\frac{\pi}{6}$

10. $\cos^{-1}\left(\cos\frac{17\pi}{14}\right)$

Note that $\frac{17\pi}{14}$ is an angle such that $\frac{17\pi}{14} > \pi$. Thus, Relationship 1 above does not apply. Since $\frac{17\pi}{14}$ is not one of our special angles, then we will not be able to find the exact value of $\cos\frac{17\pi}{14}$. Thus, we will need to use reference angles to find the answer.

The angle $\frac{17\pi}{14}$ is in the III quadrant and the reference angle for this angle is $\frac{3\pi}{14}$. Since $\cos\frac{17\pi}{14} = -\cos\frac{3\pi}{14} = \cos\frac{11\pi}{14}$, then $\cos^{-1}\left(\cos\frac{17\pi}{14}\right) = \cos^{-1}\left(\cos\frac{11\pi}{14}\right) = \frac{11\pi}{14}$ by Relationship 1above since $0 \leq \frac{11\pi}{14} \leq \pi$.

Answer: $\dfrac{11\pi}{14}$

11. $\quad Arc\cos\left(\cos\dfrac{26\pi}{15}\right)$

Note that $\dfrac{26\pi}{15}$ is an angle such that $\dfrac{26\pi}{15} > \pi$. Thus, Relationship 1 above does not apply. Since $\dfrac{26\pi}{15}$ is not one of our special angles, then we will not be able to find the exact value of $\cos\dfrac{26\pi}{15}$. Thus, we will need to use reference angles to find the answer.

The angle $\dfrac{26\pi}{15}$ is in the IV quadrant and the reference angle for this angle is $\dfrac{4\pi}{15}$. Since $\cos\dfrac{26\pi}{15} = \cos\dfrac{4\pi}{15}$, then $Arc\cos\left(\cos\dfrac{26\pi}{15}\right) =$ $Arc\cos\left(\cos\dfrac{4\pi}{15}\right) = \dfrac{4\pi}{15}$ by Relationship 1above since $0 \le \dfrac{4\pi}{15} \le \pi$.

Answer: $\dfrac{4\pi}{15}$

12. $\quad \cos^{-1}\left[\cos\left(-\dfrac{16\pi}{9}\right)\right]$

Note that $-\dfrac{16\pi}{9}$ is an angle such that $-\dfrac{16\pi}{9} < 0$. Thus, Relationship 1 above does not apply. Since $-\dfrac{16\pi}{9}$ is not one of our special angles, then we will not be able to find the exact value of $\cos\left(-\frac{16\pi}{9}\right)$. Since the angle $-\frac{16\pi}{9}$ is in the I quadrant, there are two ways to find the answer.

The first way is to use reference angles. The reference angle for the angle $-\frac{16\pi}{9}$ is $\frac{2\pi}{9}$. Since $\cos\left(-\frac{16\pi}{9}\right) = \cos\frac{2\pi}{9}$, then $\cos^{-1}\left[\cos\left(-\frac{16\pi}{9}\right)\right] = \cos^{-1}\left(\cos\frac{2\pi}{9}\right) = \frac{2\pi}{9}$ by Relationship 1 above since $0 \leq \frac{2\pi}{9} \leq \pi$.

The second way is to use a coterminal angle. Since the angle $-\frac{16\pi}{9}$ is in the I quadrant, then find the positive angle between 0 and π that is coterminal with $-\frac{16\pi}{9}$. This angle is obtained by adding 2π to $-\frac{16\pi}{9}$. Thus, $-\frac{16\pi}{9} + 2\pi = -\frac{16\pi}{9} + \frac{18\pi}{9} = \frac{2\pi}{9}$. Since the angle $-\frac{16\pi}{9}$ is coterminal with the angle $\frac{2\pi}{9}$, then $\cos\left(-\frac{16\pi}{9}\right) = \cos\frac{2\pi}{9}$. Thus, $\cos^{-1}\left[\cos\left(-\frac{16\pi}{9}\right)\right] = \cos^{-1}\left(\cos\frac{2\pi}{9}\right) = \frac{2\pi}{9}$ by Relationship 1 above since $0 \leq \frac{2\pi}{9} \leq \pi$.

NOTE: This second method will work for any negative angle, between -2π and $-\pi$, whose terminal side is in the I or II quadrant. Let θ be such a negative angle in the I or II quadrant. Then $-2\pi < \theta < -\pi$. Adding 2π to both sides of this compound inequality, we obtain that $-2\pi + 2\pi < \theta + 2\pi < -\pi + 2\pi$. Thus, $0 < \theta + 2\pi < \pi$. Since the angles θ and $\theta + 2\pi$ are coterminal, then $\cos\theta = \cos(\theta + 2\pi)$. Thus, $Arc\cos(\cos\theta) = Arc\cos[\cos(\theta + 2\pi)] = \theta + 2\pi$ by Relationship 1 above since $0 < \theta + 2\pi < \pi$.

Answer: $\dfrac{2\pi}{9}$

13. $Arc\cos\left[\cos\left(-\dfrac{7\pi}{8}\right)\right]$

Note that $-\dfrac{7\pi}{8}$ is an angle such that $-\dfrac{7\pi}{8} < 0$. Thus, Relationship 1 above does not apply. Since $-\dfrac{7\pi}{8}$ is not one of our special angles, then we will not be able to find the exact value of $\cos\left(-\dfrac{7\pi}{8}\right)$. Since the angle $-\dfrac{7\pi}{8}$ is in the III quadrant, there are two ways to find the answer.

The first way is to use reference angles. The reference angle for the angle $-\dfrac{7\pi}{8}$ is $\dfrac{\pi}{8}$. Since $\cos\left(-\dfrac{7\pi}{8}\right) = -\cos\dfrac{\pi}{8} = \cos\dfrac{7\pi}{8}$, then

$Arc\cos\left[\cos\left(-\dfrac{7\pi}{8}\right)\right] = Arc\cos\left(\cos\dfrac{7\pi}{8}\right) = \dfrac{7\pi}{8}$ by Relationship 1 above since $0 \leq \dfrac{7\pi}{8} \leq \pi$.

The second way is to use fact that the cosine function is an even function. Since the cosine function is even, then $\cos\left(-\dfrac{7\pi}{8}\right) = \cos\dfrac{7\pi}{8}$. Thus,

$Arc\cos\left[\cos\left(-\dfrac{7\pi}{8}\right)\right] = Arc\cos\left(\cos\dfrac{7\pi}{8}\right) = \dfrac{7\pi}{8}$ by Relationship 1 above since $0 \leq \dfrac{7\pi}{8} \leq \pi$.

NOTE: This second method will work for any negative angle between $-\pi$ and 0. Let θ be such a negative angle. Then $-\pi < \theta < 0$. Multiplying both sides of this compound inequality by -1, we obtain that $\pi > -\theta > 0$.

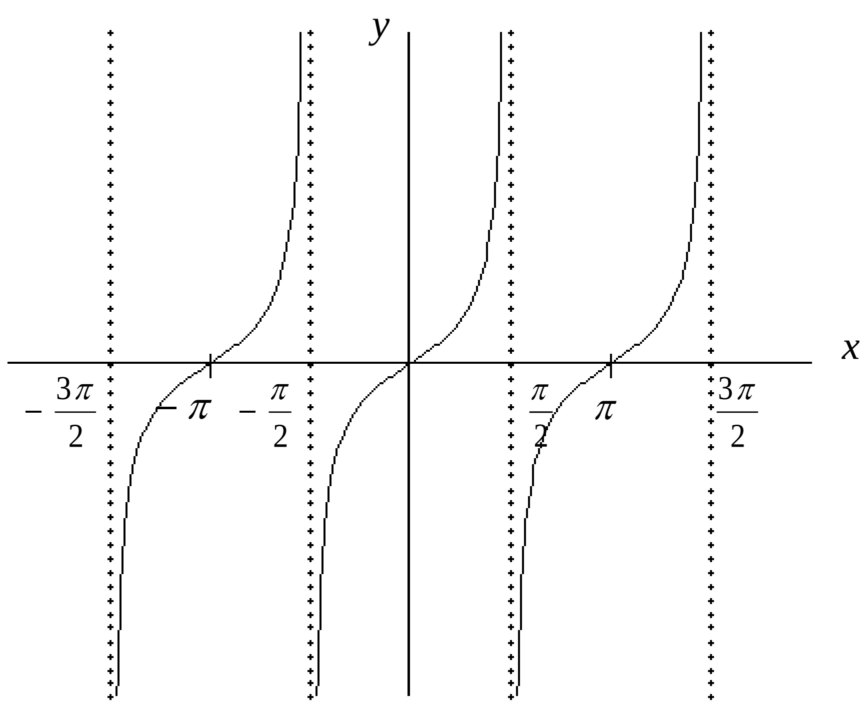
That is, $0 < -\theta < \pi$. Since the cosine function is even, then $\cos\theta = \cos(-\theta)$. Thus, $Arc\cos(\cos\theta) = Arc\cos[\cos(-\theta)] = -\theta$ by Relationship 1 above since $0 < -\theta < \pi$.

Answer: $\dfrac{7\pi}{8}$

Back to Topics List

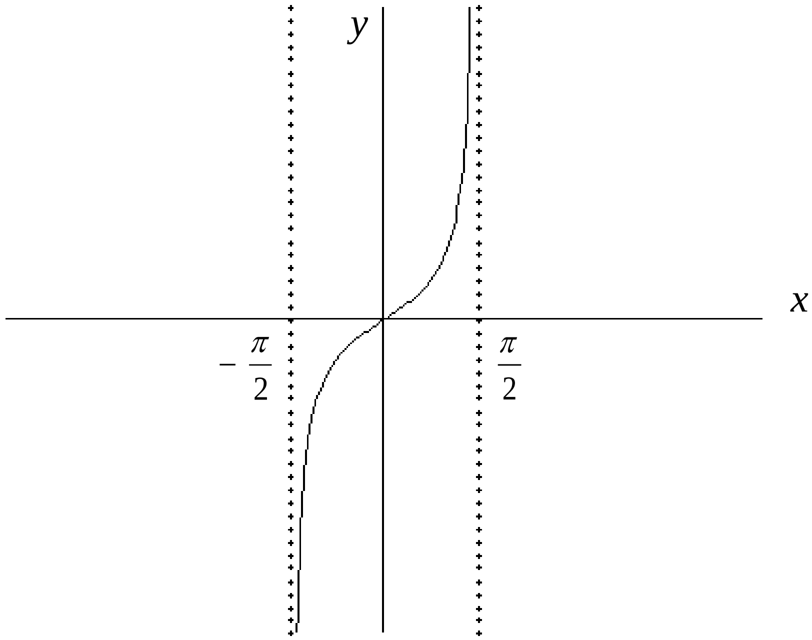
6. THE INVERSE TANGENT FUNCTION

Consider the following three cycles of the graph of $y = \tan x$:

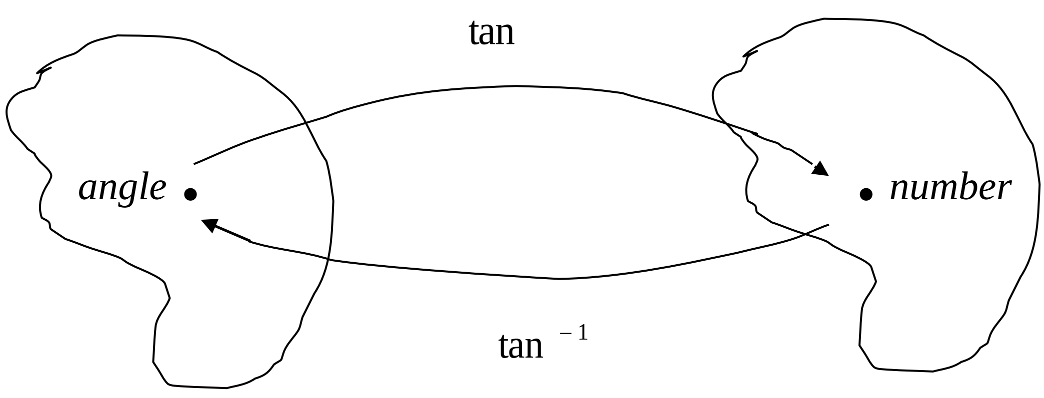


The range of $y = \tan x$ is the set of all real numbers. Thus, the domain of the inverse tangent function is the set of all real numbers. The domain of $y = \tan x$ is the set of all real numbers except for the odd integer multiples of $\frac{\pi}{2}$. However, by the horizontal line test, the function $y = \tan x$ is not one-to-one. Thus, we will need to put a restriction of the domain. We will restrict the domain of the function

$y = \tan x$ to the interval of angles $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. The graph of $y = \tan x$ on the restricted domain looks like the following:



Since the restricted domain of $y = \tan x$ is the interval of angles $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, then range of the inverse tangent function is the interval of angles $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.



Restricted domain of $\tan = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

||

Range of $\tan^{-1}$

Range of $\tan = (-\infty, \infty)$

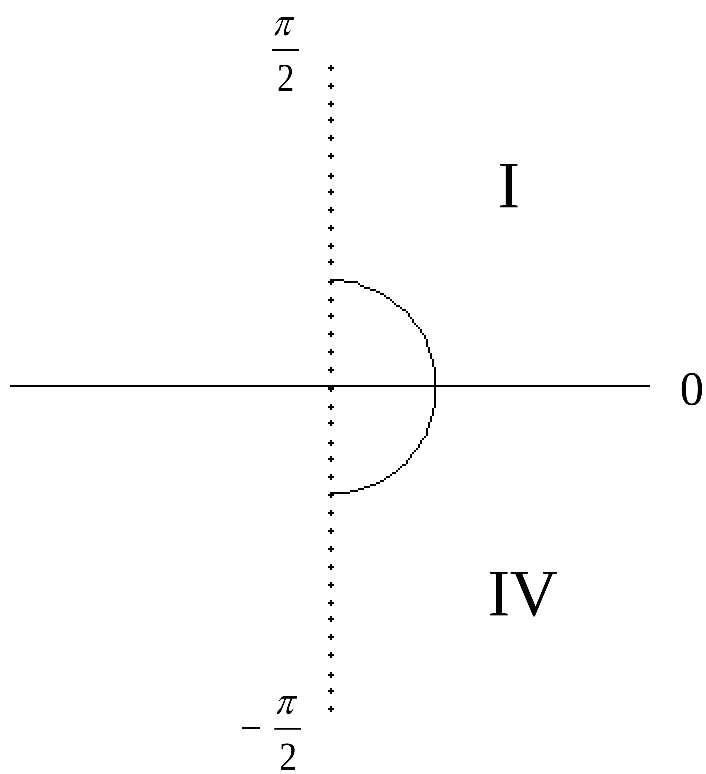
||

Domain of $\tan^{-1}$

Definition The inverse tangent function, denoted by $\tan^{-1}$, is defined by $y = \tan^{-1} x$ if and only if $\tan y = x$, where x is any real number and $-\dfrac{\pi}{2} < y < \dfrac{\pi}{2}$.

Notation: Sometimes, $\tan^{-1} x$ is denoted by $Arc\tan x$. That is, $Arc\tan x = \tan^{-1} x$.

NOTE: The angles that are in the interval $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$ are the following angles in the *xy*-plane.



Thus, the angles that are used for the inverse tangent function have their terminal side

1. In the **first quadrant** and are measured going **counterclockwise**.
2. In the **fourth quadrant** and are measured going **clockwise**.
3. On the positive *x*-axis. This is the angle of 0.

From Lesson 2, you had the following diagram to help you find the value of the tangent of the three Special Angles in the I quadrant.

$$\begin{array}{cccc}
\frac{\pi}{6}\ (30^\circ) & \frac{\pi}{4}\ (45^\circ) & \frac{\pi}{3}\ (60^\circ) & \\
| & | & | & \\
\downarrow & \downarrow & \downarrow & \text{Tangent} \\
\frac{1}{\sqrt{3}} & 1 & \sqrt{3} &
\end{array}$$

Now, reverse the arrows to have a diagram to help you find the value of the inverse tangent of these three numbers of 1, $\sqrt{3}$, and $\frac{1}{\sqrt{3}}$:

$$\begin{array}{cccc}
\frac{\pi}{6}\ (30^\circ) & \frac{\pi}{4}\ (45^\circ) & \frac{\pi}{3}\ (60^\circ) & \\
\uparrow & \uparrow & \uparrow & \\
| & | & | & \text{Inverse Tangent} \\
\frac{1}{\sqrt{3}} & 1 & \sqrt{3} &
\end{array}$$

<u>**Examples**</u> Find the exact value of the following.

1. $\tan^{-1}\sqrt{3}$

Since $\sqrt{3}$ is **positive**, then the angle answer is in the **I quadrant**. Which of the three angles of $\frac{\pi}{6}$, $\frac{\pi}{4}$, or $\frac{\pi}{3}$ would you be able to take the tangent of and get $\sqrt{3}$? Since $\sqrt{3}$ is the largest of the three numbers, which we work with for the tangent function, then the angle is the largest of these three angles. Thus, the answer is $\frac{\pi}{3}$.

There are two reasons that the answer is $\frac{\pi}{3}$:

1. $\tan\frac{\pi}{3} = \sqrt{3}$ and 2. $-\frac{\pi}{2} < \frac{\pi}{3} < \frac{\pi}{2}$

Answer: $\frac{\pi}{3}$

2. $Arc\tan\frac{1}{\sqrt{3}}$

Since $\frac{1}{\sqrt{3}}$ is **positive**, then the angle answer is in the **I quadrant**. Which of the three angles of $\frac{\pi}{6}$, $\frac{\pi}{4}$, or $\frac{\pi}{3}$ would you be able to take the tangent of and get $\frac{1}{\sqrt{3}}$? Since $\frac{1}{\sqrt{3}}$ is the smallest of the three numbers, which we work with for the tangent function, then the angle is the smallest of these three angles. Thus, the answer is $\frac{\pi}{6}$.

There are two reasons that the answer is $\frac{\pi}{6}$:

1. $\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$ and 2. $-\frac{\pi}{2} < \frac{\pi}{6} < \frac{\pi}{2}$

Answer: $\dfrac{\pi}{6}$

3. $\tan^{-1}(-1)$

Since -1 is **negative**, then the angle answer is in the **IV quadrant**. Since the angle is not in the I quadrant, let's find the **reference angle** for it.

Let $\theta = \tan^{-1}(-1)$. Then $\theta' = \tan^{-1} 1$

Which of the three angles of $\dfrac{\pi}{6}$, $\dfrac{\pi}{4}$, or $\dfrac{\pi}{3}$ would you be able to take the tangent of and get 1? Since 1 is the middle of the three numbers, which we work with for the tangent function, then the angle is the middle of these three angles. Thus, the answer is $\dfrac{\pi}{4}$.

Thus, $\theta' = \tan^{-1} 1 = \dfrac{\pi}{4}$. Now, go to the IV quadrant with this reference angle and measure the angle $\theta = \tan^{-1}(-1)$ going **clockwise**. Thus, $\tan^{-1}(-1) = -\dfrac{\pi}{4}$.

There are two reasons that the answer is $-\dfrac{\pi}{4}$:

1. $\tan\left(-\dfrac{\pi}{4}\right) = -1$ and 2. $-\dfrac{\pi}{2} < -\dfrac{\pi}{4} < \dfrac{\pi}{2}$

Answer: $-\dfrac{\pi}{4}$

4. $\quad Arc\tan\left(-\dfrac{1}{\sqrt{3}}\right)$

Since $-\dfrac{1}{\sqrt{3}}$ is **negative**, then the angle answer is in the **IV quadrant**. Since the angle is not in the I quadrant, let's find the **reference angle** for it.

Let $\theta = Arc\tan\left(-\dfrac{1}{\sqrt{3}}\right)$. Then $\theta' = Arc\tan\dfrac{1}{\sqrt{3}}$

Which of the three angles of $\dfrac{\pi}{6}$, $\dfrac{\pi}{4}$, or $\dfrac{\pi}{3}$ would you be able to take the tangent of and get $\dfrac{1}{\sqrt{3}}$? Since $\dfrac{1}{\sqrt{3}}$ is the smallest of the three numbers, which we work with for the tangent function, then the angle is the smallest of these three angles. Thus, the answer is $\dfrac{\pi}{6}$.

Thus, $\theta' = Arc\tan\dfrac{1}{\sqrt{3}} = \dfrac{\pi}{6}$. Now, go to the IV quadrant with this reference angle and measure the angle $\theta = Arc\tan\left(-\dfrac{1}{\sqrt{3}}\right)$ going **clockwise**. Thus, $Arc\tan\left(-\dfrac{1}{\sqrt{3}}\right) = -\dfrac{\pi}{6}$

There are two reasons that the answer is $-\dfrac{\pi}{6}$:

1. $\tan\left(-\dfrac{\pi}{6}\right) = -\dfrac{1}{\sqrt{3}}$ and 2. $-\dfrac{\pi}{2} < -\dfrac{\pi}{6} < \dfrac{\pi}{2}$

Answer: $-\dfrac{\pi}{6}$

5. $\tan^{-1} 0$

Since 0 is **not positive**, then the angle answer is **not** in the **I quadrant**. Since 0 is **not negative**, then the angle answer is **not** in the **IV quadrant**. Thus, the angle answer comes from one of the coordinate axis. Since there is only one axis involved for the inverse tangent function, namely, the positive *x*-axis. Thus, the answer is 0.

There are two reasons that the answer is 0:

1. $\tan 0 = 0$ and 2. $-\dfrac{\pi}{2} < 0 < \dfrac{\pi}{2}$

Answer: 0

6. $Arc\tan 1$

Since 1 is **positive**, then the angle answer is in the **I quadrant**. Which of the three angles of $\dfrac{\pi}{6}$, $\dfrac{\pi}{4}$, or $\dfrac{\pi}{3}$ would you be able to take the tangent of and get 1? Since 1 is the middle of the three numbers, which we work with for the tangent function, then the angle is the middle of these three angles. Thus, the answer is $\dfrac{\pi}{4}$.

There are two reasons that the answer is $\dfrac{\pi}{4}$:

1. $\tan \dfrac{\pi}{4} = 1$ and 2. $-\dfrac{\pi}{2} < \dfrac{\pi}{4} < \dfrac{\pi}{2}$

Answer: $\frac{\pi}{4}$

7. $\tan^{-1}(-\sqrt{3})$

Since $-\sqrt{3}$ is **negative**, then the angle answer is in the **IV quadrant**. Since the angle is not in the I quadrant, let's find the **reference angle** for it.

$$\text{Let } \theta = \tan^{-1}(-\sqrt{3}). \text{ Then } \theta' = \tan^{-1}\sqrt{3}$$

Which of the three angles of $\frac{\pi}{6}$, $\frac{\pi}{4}$, or $\frac{\pi}{3}$ would you be able to take the tangent of and get $\sqrt{3}$? Since $\sqrt{3}$ is the largest of the three numbers, which we work with for the tangent function, then the angle is the largest of these three angles. Thus, the answer is $\frac{\pi}{3}$.

Thus, $\theta' = \tan^{-1}\sqrt{3} = \frac{\pi}{3}$. Now, go to the IV quadrant with this reference angle and measure the angle $\theta = \tan^{-1}(-\sqrt{3})$ going **clockwise**. Thus, $\tan^{-1}(-\sqrt{3}) = -\frac{\pi}{3}$

There are two reasons that the answer is $-\frac{\pi}{3}$:

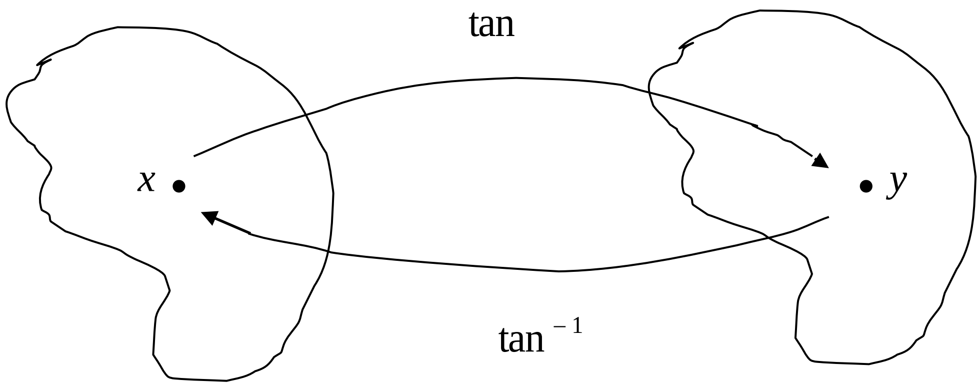
1. $\tan\left(-\frac{\pi}{3}\right) = -\sqrt{3}$ and 2. $-\frac{\pi}{2} < -\frac{\pi}{3} < \frac{\pi}{2}$

Answer: $-\frac{\pi}{3}$

Back to Topics List

7. RELATIONSHIPS BETWEEN THE TANGENT AND INVERSE TANGENT FUNCTIONS

From Section 1 above, we have the following relationships:



Restricted domain of $\tan = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ || Range of $\tan^{-1}$

Range of $\tan = (-\infty, \infty)$ || Domain of $\tan^{-1}$

We also have the following two relationships between the tangent function and its inverse tangent function:

1. $\tan^{-1}(\tan x) = x$ for all *x* in the **restricted domain of** $\tan = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

2. $\tan(\tan^{-1} y) = y$ for all *y* in the **domain of** $\tan^{-1} = (-\infty, \infty)$

<u>Examples</u> Find the exact value of the following.

1. $\tan^{-1}\left(\tan\frac{\pi}{6}\right)$

Note that $\frac{\pi}{6}$ is an angle such that $-\frac{\pi}{2} < \frac{\pi}{6} < \frac{\pi}{2}$. Thus, $\tan^{-1}\left(\tan\frac{\pi}{6}\right) = \frac{\pi}{6}$ by Relationship 1 above.

Another way to find the answer is by first finding the value of $\tan\frac{\pi}{6}$ and then finding the inverse tangent of this value: Since $\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$, then $\tan^{-1}\left(\tan\frac{\pi}{6}\right) = \tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{6}$.

Answer: $\frac{\pi}{6}$

2. $\tan[\tan^{-1}(-1)]$

Since the domain of the inverse tangent function is the set of all real numbers, then $\tan^{-1}(-1)$ is a defined angle. Thus, $\tan[\tan^{-1}(-1)] = -1$ by Relationship 2 above.

Another way to find the answer is by first finding the angle $\tan^{-1}(-1)$ and then finding the tangent of this angle: Since $\tan^{-1}(-1) = -\frac{\pi}{4}$, then $\tan[\tan^{-1}(-1)] = \tan\left(-\frac{\pi}{4}\right) = -\tan\frac{\pi}{4} = -1$.

Answer: -1

3. $Arc\tan\left[\tan\left(-\frac{3\pi}{8}\right)\right]$

Since $-\frac{3\pi}{8}$ is an angle such that $-\frac{\pi}{2} < -\frac{3\pi}{8} < \frac{\pi}{2}$, then $Arc\tan\left[\tan\left(-\frac{3\pi}{8}\right)\right] = -\frac{3\pi}{8}$ by Relationship 1 above.

Answer: $-\frac{3\pi}{8}$

4. $\tan(Arc\tan 1372)$

Since the domain of the inverse tangent function is the set of all real numbers, then $Arc\tan 1372$ is a defined angle. Thus, $\tan(Arc\tan 1372) = 1372$ by Relationship 2 above.

Answer: 1372

5. $\tan\left[\tan^{-1}\left(-\frac{5\pi}{4}\right)\right]$

Since the domain of the inverse tangent function is the set of all real numbers, then $\tan^{-1}\left(-\frac{5\pi}{4}\right)$ is a defined angle. Thus, $\tan\left[\tan^{-1}\left(-\frac{5\pi}{4}\right)\right] = -\frac{5\pi}{4}$ by Relationship 2 above.

Answer: $-\frac{5\pi}{4}$

6. $\tan^{-1}\left(\tan\frac{2\pi}{3}\right)$

Note that $\frac{2\pi}{3}$ is an angle such that $\frac{2\pi}{3} > \frac{\pi}{2}$. Thus, Relationship 1 above does not apply. There are two ways to find the answer.

The first way is to find the value of $\tan\frac{2\pi}{3}$: Since $\tan\frac{2\pi}{3} = -\tan\frac{\pi}{3} = -\sqrt{3}$, then $\tan^{-1}\left(\tan\frac{2\pi}{3}\right) = \tan^{-1}(-\sqrt{3}) = -\frac{\pi}{3}$.

The second way is to use reference angles: Since $\tan\frac{2\pi}{3} = -\tan\frac{\pi}{3} = \tan\left(-\frac{\pi}{3}\right)$, then $\tan^{-1}\left(\tan\frac{2\pi}{3}\right) = \tan^{-1}\left[\tan\left(-\frac{\pi}{3}\right)\right] = -\frac{\pi}{3}$ by Relationship 1 above since $-\frac{\pi}{2} < -\frac{\pi}{3} < \frac{\pi}{2}$.

Answer: $-\frac{\pi}{3}$

7. $Arc\tan\left[\tan\left(-\frac{5\pi}{6}\right)\right]$

Note that $-\frac{5\pi}{6}$ is an angle such that $-\frac{5\pi}{6} < -\frac{\pi}{2}$. Thus, Relationship 1 above does not apply. There are two ways to find the answer.

The first way is to find the value of $\tan\left(-\frac{5\pi}{6}\right)$: Since $\tan\left(-\frac{5\pi}{6}\right) = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$, then $Arc\tan\left[\tan\left(-\frac{5\pi}{6}\right)\right] = Arc\tan\frac{1}{\sqrt{3}} = \frac{\pi}{6}$.

The second way is to use reference angles: Since $\tan\left(-\frac{5\pi}{6}\right) = \tan\frac{\pi}{6}$, then $Arc\tan\left[\tan\left(-\frac{5\pi}{6}\right)\right] = Arc\tan\left(\tan\frac{\pi}{6}\right) = \frac{\pi}{6}$ by Relationship 1 above since $-\frac{\pi}{2} < \frac{\pi}{6} < \frac{\pi}{2}$.

Answer: $\frac{\pi}{6}$

8. $\tan^{-1}\left(\tan\frac{19\pi}{16}\right)$

Note that $\frac{19\pi}{16}$ is an angle such that $\frac{19\pi}{16} > \frac{\pi}{2}$. Thus, Relationship 1 above does not apply. Since $\frac{19\pi}{16}$ is not one of our special angles, then we will not be able to find the exact value of $\tan\frac{19\pi}{16}$. Thus, we will need to use reference angles to find the answer.

The angle $\frac{19\pi}{16}$ is in the III quadrant and the reference angle for this angle is $\frac{3\pi}{16}$. Since $\tan\frac{19\pi}{16} = \tan\frac{3\pi}{16}$, then $\tan^{-1}\left(\tan\frac{19\pi}{16}\right) =$ $\tan^{-1}\left(\tan\frac{3\pi}{16}\right) = \frac{3\pi}{16}$ by Relationship 1above since $-\frac{\pi}{2} < \frac{3\pi}{16} < \frac{\pi}{2}$.

Answer: $\frac{3\pi}{16}$

9. $Arc\tan\left(\tan\dfrac{11\pi}{12}\right)$

Note that $\dfrac{11\pi}{12}$ is an angle such that $\dfrac{11\pi}{12} > \dfrac{\pi}{2}$. Thus, Relationship 1 above does not apply. Since $\dfrac{11\pi}{12}$ is not one of our special angles, then we will not be able to find the exact value of $\tan\dfrac{11\pi}{12}$. Thus, we will need to use reference angles to find the answer.

The angle $\dfrac{11\pi}{12}$ is in the II quadrant and the reference angle for this angle is $\dfrac{\pi}{12}$. Since $\tan\dfrac{11\pi}{12} = -\tan\dfrac{\pi}{12} = \tan\left(-\dfrac{\pi}{12}\right)$, then $Arc\tan\left(\tan\dfrac{11\pi}{12}\right) =$ $Arc\tan\left[\tan\left(-\dfrac{\pi}{12}\right)\right] = -\dfrac{\pi}{12}$ by Relationship 1above since $-\dfrac{\pi}{2} < -\dfrac{\pi}{12} < \dfrac{\pi}{2}$.

Answer: $-\dfrac{\pi}{12}$

10. $\tan^{-1}\left[\tan\left(-\dfrac{30\pi}{17}\right)\right]$

Note that $-\dfrac{30\pi}{17}$ is an angle such that $-\dfrac{30\pi}{17} < -\dfrac{\pi}{2}$. Thus, Relationship 1 above does not apply. Since $-\dfrac{30\pi}{17}$ is not one of our special angles, then we will not be able to find the exact value of $\tan\left(-\frac{30\pi}{17}\right)$. Since the angle $-\frac{30\pi}{17}$ is in the I quadrant, there are two ways to find the answer.

The first way is to use reference angles. The reference angle for the angle $-\frac{30\pi}{17}$ is $\frac{4\pi}{17}$. Since $\tan\left(-\frac{30\pi}{17}\right) = \tan\frac{4\pi}{17}$, then $\tan^{-1}\left[\tan\left(-\frac{30\pi}{17}\right)\right] = \tan^{-1}\left(\tan\frac{4\pi}{17}\right) = \frac{4\pi}{17}$ by Relationship 1 above since $-\frac{\pi}{2} < \frac{4\pi}{17} < \frac{\pi}{2}$.

The second way is to use a coterminal angle. Since the angle $-\frac{30\pi}{17}$ is in the I quadrant, then find the positive angle between 0 and $\frac{\pi}{2}$ that is coterminal with $-\frac{30\pi}{17}$. This angle is obtained by adding 2π to $-\frac{30\pi}{17}$. Thus, $-\frac{30\pi}{17} + 2\pi = -\frac{30\pi}{17} + \frac{34\pi}{17} = \frac{4\pi}{17}$. Since the angle $-\frac{30\pi}{17}$ is coterminal with the angle $\frac{4\pi}{17}$, then $\tan\left(-\frac{30\pi}{17}\right) = \tan\frac{4\pi}{17}$. Thus, $\tan^{-1}\left[\tan\left(-\frac{30\pi}{17}\right)\right] = \tan^{-1}\left(\tan\frac{4\pi}{17}\right) = \frac{4\pi}{17}$ by Relationship 1 above since $-\frac{\pi}{2} < \frac{4\pi}{17} < \frac{\pi}{2}$.

NOTE: This second method will work for any negative angle, between -2π and 0, whose terminal side is in the I quadrant. Let θ be such a negative angle in the I quadrant. Since the angle θ is in the I quadrant, then $-2\pi < \theta < -\frac{3\pi}{2}$. Adding 2π to both sides of this compound inequality, we obtain that $-2\pi + 2\pi < \theta + 2\pi < -\frac{3\pi}{2} + 2\pi$. Thus,

$0 < \theta + 2\pi < \dfrac{\pi}{2}$. Since the angles θ and $\theta + 2\pi$ are coterminal, then $\tan\theta = \tan(\theta + 2\pi)$. Thus, $Arc\tan(\tan\theta) = Arc\tan[\tan(\theta + 2\pi)] = \theta + 2\pi$ by Relationship 1 above since $0 < \theta + 2\pi < \dfrac{\pi}{2}$.

As we will see in our last example, this second method will also work for a positive angle, between 0 and 2π, whose terminal side is in the IV quadrant. However, this second method will not work for any angle, between -2π and 2π, whose terminal side is in either the II or III quadrant.

Answer: $\dfrac{4\pi}{17}$

11. $Arc\tan\left(\tan\dfrac{32\pi}{19}\right)$

Note that $\dfrac{32\pi}{19}$ is an angle such that $\dfrac{32\pi}{19} > \dfrac{\pi}{2}$. Thus, Relationship 1 above does not apply. Since $\dfrac{32\pi}{19}$ is not one of our special angles, then we will not be able to find the exact value of $\tan\dfrac{32\pi}{19}$. Since the angle $\dfrac{32\pi}{19}$ is in the IV quadrant, there are two ways to find the answer.

The first way is to use reference angles. The reference angle for the angle $\dfrac{32\pi}{19}$ is $\dfrac{6\pi}{19}$. Since $\tan\dfrac{32\pi}{19} = -\tan\dfrac{6\pi}{19} = \tan\left(-\dfrac{6\pi}{19}\right)$, then $Arc\tan\left(\tan\dfrac{32\pi}{19}\right) = Arc\tan\left[\tan\left(-\dfrac{6\pi}{19}\right)\right] = -\dfrac{6\pi}{19}$ by Relationship 1 above since $-\dfrac{\pi}{2} < -\dfrac{6\pi}{19} < \dfrac{\pi}{2}$.

The second way is to use a coterminal angle. Since the angle $\frac{32\pi}{19}$ is in the IV quadrant, then find the negative angle between $-\frac{\pi}{2}$ and 0 that is coterminal with $\frac{32\pi}{19}$. This angle is obtained by subtracting 2π from $\frac{32\pi}{19}$. Thus, $\frac{32\pi}{19} - 2\pi = \frac{32\pi}{19} - \frac{38\pi}{19} = -\frac{6\pi}{19}$. Since the angle $\frac{32\pi}{19}$ is coterminal with the angle $-\frac{6\pi}{19}$, then $\tan\frac{32\pi}{19} = \tan\left(-\frac{6\pi}{19}\right)$. Thus, $Arc\tan\left(\tan\frac{32\pi}{19}\right) = Arc\tan\left[\tan\left(-\frac{6\pi}{19}\right)\right] = -\frac{6\pi}{19}$ by Relationship 1 above since $-\frac{\pi}{2} < -\frac{6\pi}{19} < \frac{\pi}{2}$.

NOTE: This second method will work for any positive angle, between 0 and 2π, whose terminal side is in the IV quadrant. Let θ be such a positive angle in the IV quadrant. Since the angle θ is in the IV quadrant, then $\frac{3\pi}{2} < \theta < 2\pi$. Subtracting 2π from both sides of this compound inequality, we obtain that $\frac{3\pi}{2} - 2\pi < \theta - 2\pi < 2\pi - 2\pi$. Thus, $-\frac{\pi}{2} < \theta - 2\pi < 0$. Since the angles θ and $\theta - 2\pi$ are coterminal, then $\tan\theta = \tan(\theta - 2\pi)$. Thus, $Arc\tan(\tan\theta) = Arc\tan[\tan(\theta - 2\pi)] = \theta - 2\pi$ by Relationship 1 above since $-\frac{\pi}{2} < \theta - 2\pi < 0$.

Again, this second method will not work for any angle, between -2π and 2π, whose terminal side is in either the II or III quadrant.

Answer: $-\frac{6\pi}{19}$

Back to Topics List

8. APPLICATIONS OF THE INVERSE TRIGONOMETRIC FUNCTIONS

Examples Find the exact and the approximate angle θ , which is between $0°$ and $360\,°$, which passes through the following points.

1. $(-8, 6)$

From Lesson 5, we have that $\tan\theta = \dfrac{y}{x} = \dfrac{6}{-8} = -\dfrac{3}{4}$. The most common **mistake** made students at this point is to say that $\theta = \tan^{-1}\left(-\dfrac{3}{4}\right)$. Of course, this is **clearly** not true. Since the point $(-8, 6)$ is in the II quadrant and the terminal side of the angle θ passes through this point, then the terminal side of θ is in the II quadrant. However, since $-\dfrac{3}{4}$ is **negative**, then the angle $\tan^{-1}\left(-\dfrac{3}{4}\right)$ is in the **IV quadrant**. Thus,

$$\theta \neq \tan^{-1}\left(-\frac{3}{4}\right).$$

Use the inverse tangent function to find the reference angle for the angle θ:

$$\tan\theta = -\frac{3}{4} \Rightarrow \tan\theta' = \frac{3}{4} \Rightarrow \theta' = \tan^{-1}\frac{3}{4} \approx 36.9\,°$$

Now, go to the II quadrant and measure the angle θ.

Exact: $\theta = 180\,° - \theta' = 180\,° - \tan^{-1}\dfrac{3}{4}$

Approximate: $\theta = 180\,° - \theta' \approx 180\,° - 36.9\,° = 143.1\,°$

NOTE: Any angle coterminal to this angle will also pass through the point $(-8, 6)$. However, these coterminal angles would not be between $0°$ and $360°$.

Answers: $\theta = 180° - \tan^{-1}\dfrac{3}{4}$; $\theta \approx 143.1°$

2. $(\sqrt{3}, -9)$

From Lesson 5, we have that $\tan\theta = \dfrac{y}{x} = \dfrac{-9}{\sqrt{3}} = -\dfrac{9\sqrt{3}}{3} = -3\sqrt{3}$. For this problem, it is almost true that $\theta = \tan^{-1}(-3\sqrt{3})$. The point $(\sqrt{3}, -9)$ is in the IV quadrant. Since $-3\sqrt{3}$ is **negative**, then the angle $\tan^{-1}(-3\sqrt{3})$ is in the **IV quadrant**. However, the angle $\tan^{-1}(-3\sqrt{3})$ is negative. The angle θ is to be between $0°$ and $360°$. Thus,

$$\theta \neq \tan^{-1}(-3\sqrt{3}).$$

Use the inverse tangent function to find the reference angle for the angle θ:

$$\tan\theta = -3\sqrt{3} \Rightarrow \tan\theta' = 3\sqrt{3} \Rightarrow \theta' = \tan^{-1} 3\sqrt{3} \approx 79.1°$$

Now, go to the IV quadrant and measure the angle θ.

Exact: $\theta = 360° - \theta' = 360° - \tan^{-1} 3\sqrt{3}$

Approximate: $\theta = 360° - \theta' \approx 360° - 79.1° = 280.9°$

NOTE: Any angle coterminal to this angle will also pass through the point $(\sqrt{3}, -9)$. However, these coterminal angles would not be between $0°$ and $360°$.

Answers: $\theta = 360° - \tan^{-1} 3\sqrt{3}$; $\theta \approx 280.9°$

3. $(-12, -18)$

From Lesson 5, we have that $\tan\theta = \dfrac{y}{x} = \dfrac{-18}{-12} = \dfrac{3}{2}$. Since the point $(-12, -18)$ is in the III quadrant and the terminal side of the angle θ passes through this point, then the terminal side of θ is in the III quadrant. However, since $\dfrac{3}{2}$ is **positive**, then the angle $\tan^{-1}\dfrac{3}{2}$ is in the **I quadrant**. Thus,

$$\theta \neq \tan^{-1}\frac{3}{2}.$$

Use the inverse tangent function to find the reference angle for the angle θ:

$$\tan\theta = \frac{3}{2} \Rightarrow \tan\theta' = \frac{3}{2} \Rightarrow \theta' = \tan^{-1}\frac{3}{2} \approx 56.3^\circ$$

Now, go to the III quadrant and measure the angle θ.

Exact: $\theta = 180^\circ + \theta' = 180^\circ + \tan^{-1}\dfrac{3}{2}$

Approximate: $\theta = 180^\circ + \theta' \approx 180^\circ + 56.3^\circ = 236.3^\circ$

NOTE: Any angle coterminal to this angle will also pass through the point $(-12, -18)$. However, these coterminal angles would not be between 0° and 360°.

Answers: $\theta = 180^\circ + \tan^{-1}\dfrac{3}{2}$; $\theta \approx 236.3^\circ$

4. $(7, 3)$

From Lesson 5, we have that $\tan\theta = \dfrac{y}{x} = \dfrac{3}{7}$. Since the point $(7, 3)$ is in the **I quadrant**, then for this problem, it is true that $\theta = \tan^{-1}\dfrac{3}{7}$. Thus,

$$\theta = \tan^{-1}\frac{3}{7} \approx 23.2^\circ.$$

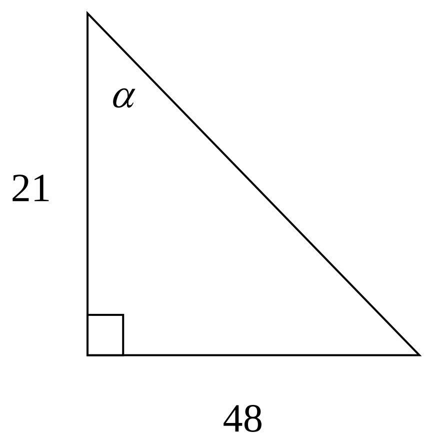
Exact: $\theta = \tan^{-1}\dfrac{3}{7}$

Approximate: $\theta \approx 23.2^\circ$

NOTE: Any angle coterminal to this angle will also pass through the point $(7, 3)$. However, these coterminal angles would not be between 0° and 360°.

Answers: $\theta = \tan^{-1}\dfrac{3}{7}$; $\theta \approx 23.2^\circ$

Example Find the exact and the approximate value (in degrees) for α.



Notice in the right triangle, 21 is the adjacent side of the given angle α and the given value of 48 is the opposite side of the given angle α. Restricting to the cosine, sine, and tangent functions, which one of these three functions involves the opposite and adjacent sides of the angle? Answer: The tangent function. Thus, we have that

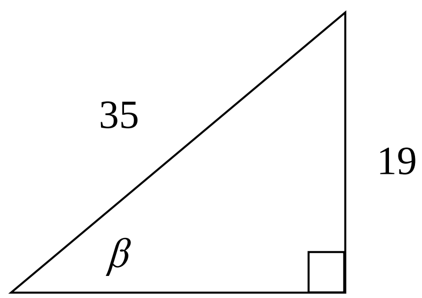
$$\tan\alpha = \frac{48}{21} = \frac{16}{7}$$

Since α is an acute angle, then α is in the **I quadrant**. Thus,

$$\alpha = \tan^{-1}\frac{16}{7} \approx 66.4°$$

Answer: $\alpha = \tan^{-1}\frac{16}{7}$; $\alpha \approx 66.4°$

<u>Example</u> Find the exact and the approximate value (in degrees) for β.



Notice in the right triangle, 19 is the opposite side of the given angle β and the given value of 35 is the hypotenuse of the right triangle. Restricting to the cosine, sine, and tangent functions, which one of these three functions involves the opposite side of the angle and the hypotenuse? Answer: The sine function. Thus, we have that

$$\sin\beta = \frac{19}{35}$$

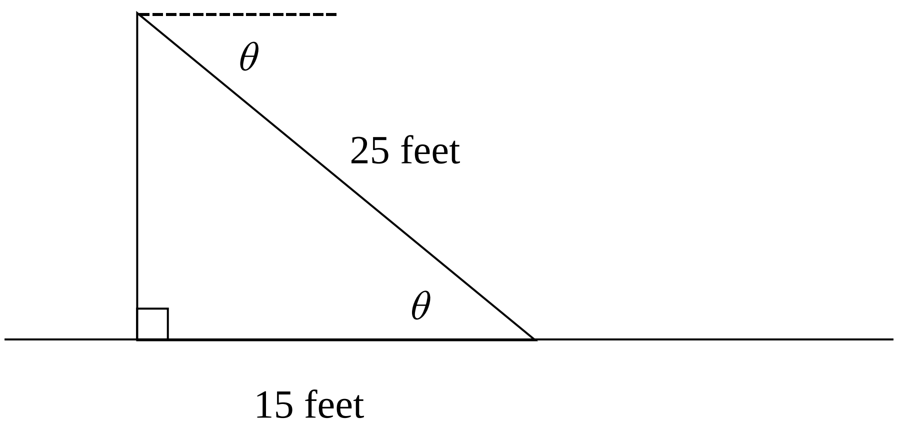
Since β is an acute angle, then β is in the **I quadrant**. Thus,

$$\beta = \sin^{-1}\frac{19}{35} \approx 32.9°$$

Answer: $\beta = \sin^{-1}\dfrac{19}{35}$; $\beta \approx 32.9°$

Example A 25-foot ladder is leaning against the top of a vertical wall. If the bottom of the ladder is 15 feet from the base of the wall, then find the angle of depression from the top of the ladder to the ground.

Let θ be the angle of depression.



Notice in the right triangle, 15 is the adjacent side of the given angle θ and the given value of 25 is the hypotenuse of the right triangle. Restricting to the cosine, sine, and tangent functions, which one of these three functions involves the opposite side of the angle and the hypotenuse? Answer: The cosine function. Thus, we have that

$$\cos\theta = \frac{15}{25} = \frac{3}{5}$$

Since θ is an acute angle, then θ is in the **I quadrant**. Thus,

$$\theta = \cos^{-1}\frac{3}{5} \approx 53.1°$$

Answer: $\theta = \cos^{-1}\dfrac{3}{5}$; $\theta \approx 53.1°$

Examples Find the exact value of the following.

1. $\cos\left(\sin^{-1}\frac{2}{5}\right)$

We know that $\sin^{-1}\frac{2}{5}$ is an angle. Let's give another name to this angle.

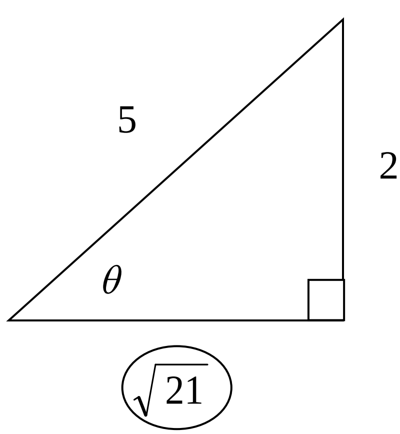
Let $\theta = \sin^{-1}\frac{2}{5}$. Thus, $\cos\left(\sin^{-1}\frac{2}{5}\right) = \cos\theta$.

Thus, to find the exact value of $\cos\left(\sin^{-1}\frac{2}{5}\right)$, we only need to find the exact value of $\cos\theta$. To find the exact value of $\cos\theta$, we need to know what quadrant the angle θ is in and we need to know the exact value of $\cos\theta'$.

Since $\theta = \sin^{-1}\frac{2}{5}$, then θ is in the **I quadrant** since the $\frac{2}{5}$ is **positive**. Since θ is in the I quadrant, then $\cos\theta$ is positive.

Also, since the angle θ is in the I quadrant, then it's an acute angle. Thus, it could be put into a right triangle.

Since $\theta = \sin^{-1}\frac{2}{5}$, then $\sin\theta = \frac{2}{5} = \frac{opp}{hyp}$



Thus, $\cos\theta = \dfrac{adj}{hyp} = \dfrac{\sqrt{21}}{5}$.

Thus, $\cos\left(\sin^{-1}\dfrac{2}{5}\right) = \cos\theta = \dfrac{\sqrt{21}}{5}$

Answer: $\dfrac{\sqrt{21}}{5}$

NOTE:

$\sin\left(\sin^{-1}\dfrac{2}{5}\right) = \sin\theta = \dfrac{2}{5}$ $\qquad$ $\csc\left(\sin^{-1}\dfrac{2}{5}\right) = \csc\theta = \dfrac{5}{2}$

$\sec\left(\sin^{-1}\dfrac{2}{5}\right) = \sec\theta = \dfrac{5}{\sqrt{21}}$

$\tan\left(\sin^{-1}\dfrac{2}{5}\right) = \tan\theta = \dfrac{2}{\sqrt{21}}$ $\qquad$ $\cot\left(\sin^{-1}\dfrac{2}{5}\right) = \cot\theta = \dfrac{\sqrt{21}}{2}$

Create your own example: If the angle is $\sin^{-1}\dfrac{a}{b}$, where the numbers a and b are positive and $a < b$, then

$\cos\left(\sin^{-1}\dfrac{a}{b}\right) = \dfrac{\sqrt{b^2 - a^2}}{b}$ $\qquad$ $\sec\left(\sin^{-1}\dfrac{a}{b}\right) = \dfrac{b}{\sqrt{b^2 - a^2}}$

$\sin\left(\sin^{-1}\dfrac{a}{b}\right) = \dfrac{a}{b}$ $\qquad$ $\csc\left(\sin^{-1}\dfrac{a}{b}\right) = \dfrac{b}{a}$

$$\tan\left(\sin^{-1}\frac{a}{b}\right) = \frac{a}{\sqrt{b^2 - a^2}} \qquad \cot\left(\sin^{-1}\frac{a}{b}\right) = \frac{\sqrt{b^2 - a^2}}{a}$$

NOTE: Since a and b are positive numbers, then the number $\frac{a}{b}$ is **positive**. Thus, the angle $\sin^{-1}\frac{a}{b}$ is in the **I quadrant**.

2. $\cot\left[Arc\cos\left(-\frac{\sqrt{33}}{7}\right)\right]$

Let's give another name to the angle $Arc\cos\left(-\frac{\sqrt{33}}{7}\right)$.

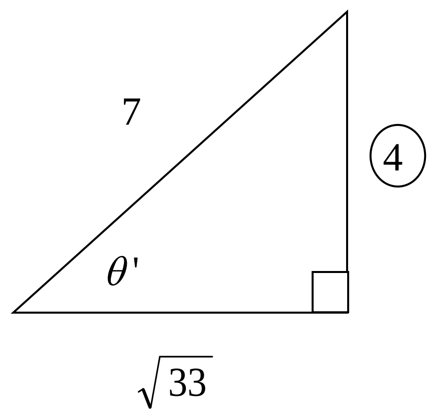
Let $\theta = Arc\cos\left(-\frac{\sqrt{33}}{7}\right)$. Thus, $\cot\left[Arc\cos\left(-\frac{\sqrt{33}}{7}\right)\right] = \cot\theta$.

Thus, to find the exact value of $\cot\left[Arc\cos\left(-\frac{\sqrt{33}}{7}\right)\right]$, we only need to find the exact value of $\cot\theta$. To find the exact value of $\cot\theta$, we can find the exact of $\tan\theta$ and then take the reciprocal of this value. To find the exact value of $\tan\theta$ we need to know what quadrant the angle θ is in and we need to know the exact value of $\tan\theta'$.

Since $\theta = Arc\cos\left(-\frac{\sqrt{33}}{7}\right)$, then θ is in the **II quadrant** since the $-\frac{\sqrt{33}}{7}$ is **negative**. Since θ is in the II quadrant, then $\tan\theta$ is negative.

Since the angle θ is in the II quadrant, then it is **not** an acute angle. Thus, it can **not** be put into a right triangle. However, the reference angle θ' of the angle θ can be put into a right triangle.

$$\theta = Arc\cos\left(-\frac{\sqrt{33}}{7}\right) \Rightarrow \cos\theta = -\frac{\sqrt{33}}{7} \Rightarrow \cos\theta' = \frac{\sqrt{33}}{7} = \frac{adj}{hyp}$$



Thus, $\tan\theta' = \dfrac{opp}{adj} = \dfrac{4}{\sqrt{33}}$. Thus, $\tan\theta = -\tan\theta' = -\dfrac{4}{\sqrt{33}}$.

Thus, $\cot\left[Arc\cos\left(-\dfrac{\sqrt{33}}{7}\right)\right] = \cot\theta = -\dfrac{\sqrt{33}}{4}$

Answer: $-\dfrac{\sqrt{33}}{4}$

NOTE:

$$\cos\left[Arc\cos\left(-\frac{\sqrt{33}}{7}\right)\right] = \cos\theta = -\cos\theta' = -\frac{\sqrt{33}}{7}$$

$$\sec\left[Arc\cos\left(-\frac{\sqrt{33}}{7}\right)\right] = \sec\theta = -\frac{7}{\sqrt{33}}$$

$$\sin\left[Arc\cos\left(-\frac{\sqrt{33}}{7} \right) \right] = \sin\theta = \sin\theta' = \frac{4}{7}$$

$$\csc\left[Arc\cos\left(-\frac{\sqrt{33}}{7} \right) \right] = \csc\theta = \frac{7}{4}$$

$$\tan\left[Arc\cos\left(-\frac{\sqrt{33}}{7} \right) \right] = \tan\theta = -\frac{4}{\sqrt{33}}$$

Create your own example: If the angle is $Arc\cos\left(-\frac{a}{b} \right)$, where the numbers a and b are positive and $a < b$, then

$$\cos\left[Arc\cos\left(-\frac{a}{b} \right) \right] = -\frac{a}{b} \qquad \sec\left[Arc\cos\left(-\frac{a}{b} \right) \right] = -\frac{b}{a}$$

$$\sin\left[Arc\cos\left(-\frac{a}{b} \right) \right] = \frac{\sqrt{b^2 - a^2}}{b} \qquad \csc\left[Arc\cos\left(-\frac{a}{b} \right) \right] = \frac{b}{\sqrt{b^2 - a^2}}$$

$$\tan\left[Arc\cos\left(-\frac{a}{b} \right) \right] = -\frac{\sqrt{b^2 - a^2}}{a} \qquad \cot\left[Arc\cos\left(-\frac{a}{b} \right) \right] = -\frac{a}{\sqrt{b^2 - a^2}}$$

NOTE: Since a and b are positive numbers, then the number $-\frac{a}{b}$ is **negative**. Thus, the angle $Arc\cos\left(-\frac{a}{b} \right)$ is in the **II quadrant**.

3. $\sec\left[\tan^{-1}\left(-\frac{8}{3} \right) \right]$

Let's give another name to the angle $\tan^{-1}\left(-\frac{8}{3}\right)$.

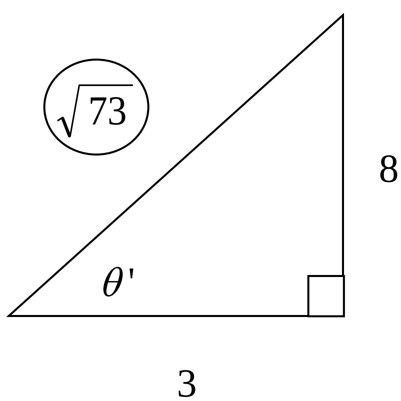
$$\text{Let } \theta = \tan^{-1}\left(-\frac{8}{3}\right). \text{ Thus, } \sec\left[\tan^{-1}\left(-\frac{8}{3}\right)\right] = \sec\theta.$$

Thus, to find the exact value of $\sec\left[\tan^{-1}\left(-\frac{8}{3}\right)\right]$, we only need to find the exact value of $\sec\theta$. To find the exact value of $\sec\theta$, we can find the exact of $\cos\theta$ and then take the reciprocal of this value. To find the exact value of $\cos\theta$ we need to know what quadrant the angle θ is in and we need to know the exact value of $\cos\theta'$.

Since $\theta = \tan^{-1}\left(-\frac{8}{3}\right)$, then θ is in the **IV quadrant** since the $-\frac{8}{3}$ is **negative**. Since θ is in the IV quadrant, then $\cos\theta$ is positive.

Since the angle θ is in the IV quadrant, then it is **not** an acute angle. Thus, it can **not** be put into a right triangle. However, the reference angle θ' of the angle θ can be put into a right triangle.

$$\theta = \tan^{-1}\left(-\frac{8}{3}\right) \Rightarrow \tan\theta = -\frac{8}{3} \Rightarrow \tan\theta' = \frac{8}{3} = \frac{opp}{adj}$$



Thus, $\cos\theta' = \dfrac{adj}{hyp} = \dfrac{3}{\sqrt{73}}$. Thus, $\cos\theta = \cos\theta' = \dfrac{3}{\sqrt{73}}$.

Thus, $\sec\left[\tan^{-1}\left(-\dfrac{8}{3}\right)\right] = \sec\theta = \dfrac{\sqrt{73}}{3}$

Answer: $\dfrac{\sqrt{73}}{3}$

NOTE:

$$\cos\left[\tan^{-1}\left(-\frac{8}{3}\right)\right] = \cos\theta = \frac{3}{\sqrt{73}}$$

$$\sin\left[\tan^{-1}\left(-\frac{8}{3}\right)\right] = \sin\theta = -\sin\theta' = -\frac{8}{\sqrt{73}}$$

$$\csc\left[\tan^{-1}\left(-\frac{8}{3}\right)\right] = \csc\theta = -\frac{\sqrt{73}}{8}$$

$$\tan\left[\tan^{-1}\left(-\frac{8}{3}\right)\right] = \tan\theta = -\frac{8}{3}$$

$$\cot\left[\tan^{-1}\left(-\frac{8}{3}\right)\right] = \cot\theta = -\frac{3}{8}$$

Create your own example: If the angle is $\tan^{-1}\left(-\dfrac{a}{b}\right)$, where the numbers a and b are positive, then

$$\cos\left[\tan^{-1}\left(-\frac{a}{b}\right)\right] = \frac{b}{\sqrt{a^2 + b^2}} \qquad \sec\left[\tan^{-1}\left(-\frac{a}{b}\right)\right] = \frac{\sqrt{a^2 + b^2}}{b}$$

$$\sin\left[\tan^{-1}\left(-\frac{a}{b}\right)\right] = -\frac{a}{\sqrt{a^2 + b^2}} \qquad \csc\left[\tan^{-1}\left(-\frac{a}{b}\right)\right] = -\frac{\sqrt{a^2 + b^2}}{a}$$

$$\tan\left[\tan^{-1}\left(-\frac{a}{b}\right)\right] = -\frac{a}{b} \qquad \cot\left[\tan^{-1}\left(-\frac{a}{b}\right)\right] = -\frac{b}{a}$$

NOTE: Since a and b are positive numbers, then the number $-\frac{a}{b}$ is **negative**. Thus, the angle $\tan^{-1}\left(-\frac{a}{b}\right)$ is in the **IV quadrant**.

4. $\csc\left(\cos^{-1}\frac{\pi}{6}\right)$

Let's give another name to the angle $\cos^{-1}\frac{\pi}{6}$.

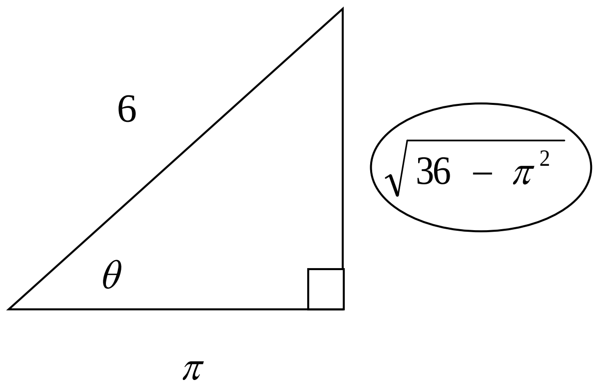
Let $\theta = \cos^{-1}\frac{\pi}{6}$. Thus, $\csc\left(\cos^{-1}\frac{\pi}{6}\right) = \csc\theta$.

Thus, to find the exact value of $\csc\left(\cos^{-1}\frac{\pi}{6}\right)$, we only need to find the exact value of $\csc\theta$. To find the exact value of $\csc\theta$, we can find the exact of $\sin\theta$ and then take the reciprocal of this value. To find the exact value of $\sin\theta$ we need to know what quadrant the angle θ is in and we need to know the exact value of $\sin\theta'$.

Since $\theta = \cos^{-1}\frac{\pi}{6}$, then θ is in the **I quadrant** since the $\frac{\pi}{6}$ is **positive**. Since θ is in the I quadrant, then $\sin\theta$ is positive.

Also, since the angle θ is in the I quadrant, then it's an acute angle. Thus, it could be put into a right triangle.

Since $\theta = \cos^{-1}\frac{\pi}{6}$, then $\cos\theta = \frac{\pi}{6} = \frac{adj}{hyp}$



Thus, $\sin\theta = \frac{opp}{hyp} = \frac{\sqrt{36 - \pi^2}}{6}$.

Thus, $\csc\left(\cos^{-1}\frac{\pi}{6}\right) = \csc\theta = \frac{6}{\sqrt{36 - \pi^2}}$

Answer: $\frac{6}{\sqrt{36 - \pi^2}}$

NOTE:

$\cos\left(\cos^{-1}\frac{\pi}{6}\right) = \cos\theta = \frac{\pi}{6}$

$\sec\left(\cos^{-1}\frac{\pi}{6}\right) = \sec\theta = \frac{6}{\pi}$

$\sin\left(\cos^{-1}\frac{\pi}{6}\right) = \sin\theta = \frac{\sqrt{36 - \pi^2}}{6}$

$\tan\left(\cos^{-1}\frac{\pi}{6}\right) = \tan\theta = \frac{\sqrt{36 - \pi^2}}{\pi}$

$$\cot\left(\cos^{-1}\frac{\pi}{6}\right) = \cot\theta = \frac{\pi}{\sqrt{36 - \pi^2}}$$

Create your own example: If the angle is $\cos^{-1}\dfrac{a}{b}$, where the numbers *a* and *b* are positive and $a < b$, then

$$\cos\left(\cos^{-1}\frac{a}{b}\right) = \frac{a}{b} \qquad \sec\left(\cos^{-1}\frac{a}{b}\right) = \frac{b}{a}$$

$$\sin\left(\cos^{-1}\frac{a}{b}\right) = \frac{\sqrt{b^2 - a^2}}{b} \qquad \csc\left(\cos^{-1}\frac{a}{b}\right) = \frac{b}{\sqrt{b^2 - a^2}}$$

$$\tan\left(\cos^{-1}\frac{a}{b}\right) = \frac{\sqrt{b^2 - a^2}}{a} \qquad \cot\left(\cos^{-1}\frac{a}{b}\right) = \frac{a}{\sqrt{b^2 - a^2}}$$

NOTE: Since *a* and *b* are positive numbers, then the number $\dfrac{a}{b}$ is **positive**. Thus, the angle $\cos^{-1}\dfrac{a}{b}$ is in the **I quadrant**.

5. $\tan\left[\sin^{-1}\left(-\dfrac{8}{17}\right)\right]$

Let's give another name to the angle $\sin^{-1}\left(-\dfrac{8}{17}\right)$.

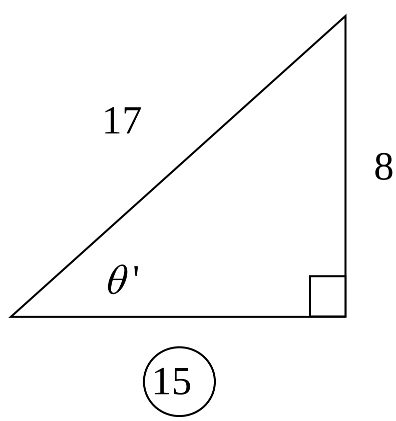
Let $\theta = \sin^{-1}\left(-\dfrac{8}{17}\right)$. Thus, $\tan\left[\sin^{-1}\left(-\dfrac{8}{17}\right)\right] = \tan\theta$.

Thus, to find the exact value of $\tan\left[\sin^{-1}\left(-\frac{8}{17}\right)\right]$, we only need to find the exact value of $\tan\theta$. To find the exact value of $\tan\theta$ we need to know what quadrant the angle θ is in and we need to know the exact value of $\tan\theta'$.

Since $\theta = \sin^{-1}\left(-\frac{8}{17}\right)$, then θ is in the **IV quadrant** since the $-\frac{8}{17}$ is **negative**. Since θ is in the IV quadrant, then $\tan\theta$ is negative.

Since the angle θ is in the IV quadrant, then it is **not** an acute angle. Thus, it can **not** be put into a right triangle. However, the reference angle θ' of the angle θ can be put into a right triangle.

$$\theta = \sin^{-1}\left(-\frac{8}{17}\right) \Rightarrow \sin\theta = -\frac{8}{17} \Rightarrow \sin\theta' = \frac{8}{17} = \frac{opp}{hyp}$$



Thus, $\tan\theta' = \frac{opp}{adj} = \frac{8}{15}$. Thus, $\tan\theta = -\tan\theta' = -\frac{8}{15}$.

Thus, $\tan\left[\sin^{-1}\left(-\frac{8}{17}\right)\right] = \tan\theta = -\frac{8}{15}$

Answer: $-\frac{8}{15}$

NOTE:

$$\cos\left[\sin^{-1}\left(-\frac{8}{17}\right)\right] = \cos\theta = \frac{15}{17}$$

$$\sec\left[\sin^{-1}\left(-\frac{8}{17}\right)\right] = \sec\theta = \frac{17}{15}$$

$$\sin\left[\sin^{-1}\left(-\frac{8}{17}\right)\right] = \sin\theta = -\frac{8}{17}$$

$$\csc\left[\sin^{-1}\left(-\frac{8}{17}\right)\right] = \csc\theta = -\frac{17}{8}$$

$$\cot\left[\sin^{-1}\left(-\frac{8}{17}\right)\right] = \cot\theta = -\frac{15}{8}$$

Create your own example: If the angle is $\sin^{-1}\left(-\frac{a}{b}\right)$, where the numbers a and b are positive and $a < b$, then

$$\cos\left[\sin^{-1}\left(-\frac{a}{b}\right)\right] = \frac{\sqrt{b^2 - a^2}}{b} \qquad \sec\left[\sin^{-1}\left(-\frac{a}{b}\right)\right] = \frac{b}{\sqrt{b^2 - a^2}}$$

$$\sin\left[\sin^{-1}\left(-\frac{a}{b}\right)\right] = -\frac{a}{b} \qquad \csc\left[\sin^{-1}\left(-\frac{a}{b}\right)\right] = -\frac{b}{a}$$

$$\tan\left[\sin^{-1}\left(-\frac{a}{b}\right)\right] = -\frac{a}{\sqrt{b^2 - a^2}} \qquad \cot\left[\sin^{-1}\left(-\frac{a}{b}\right)\right] = -\frac{\sqrt{b^2 - a^2}}{a}$$

NOTE: Since a and b are positive numbers, then the number $-\dfrac{a}{b}$ is **negative**. Thus, the angle $\sin^{-1}\left(-\dfrac{a}{b}\right)$ is in the **IV quadrant**.

6. $\sin\left(Arc\tan\dfrac{\sqrt{6}}{9}\right)$

Let's give another name to the angle $Arc\tan\dfrac{\sqrt{6}}{9}$.

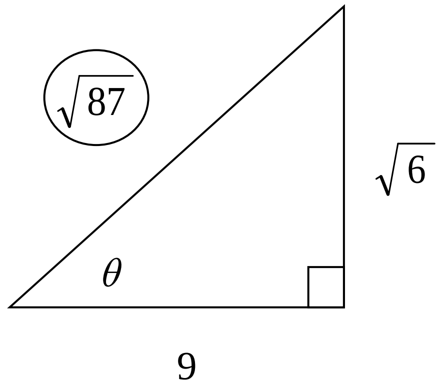
Let $\theta = Arc\tan\dfrac{\sqrt{6}}{9}$. Thus, $\sin\left(Arc\tan\dfrac{\sqrt{6}}{9}\right) = \sin\theta$.

Thus, to find the exact value of $\sin\left(Arc\tan\dfrac{\sqrt{6}}{9}\right)$, we only need to find the exact value of $\sin\theta$. To find the exact value of $\sin\theta$, we need to know what quadrant the angle θ is in and we need to know the exact value of $\sin\theta'$.

Since $\theta = Arc\tan\dfrac{\sqrt{6}}{9}$, then θ is in the **I quadrant** since the $\dfrac{\sqrt{6}}{9}$ is **positive**. Since θ is in the I quadrant, then $\sin\theta$ is positive.

Also, since the angle θ is in the I quadrant, then it's an acute angle. Thus, it could be put into a right triangle.

Since $\theta = Arc\tan\dfrac{\sqrt{6}}{9}$, then $\tan\theta = \dfrac{\sqrt{6}}{9} = \dfrac{opp}{adj}$



Thus, $\sin\,\theta = \dfrac{opp}{hyp} = \dfrac{\sqrt{6}}{\sqrt{87}} = \sqrt{\dfrac{6}{87}} = \sqrt{\dfrac{2}{29}} = \dfrac{\sqrt{2}}{\sqrt{29}}$.

Thus, $\sin\left(Arc\tan\dfrac{\sqrt{6}}{9}\right) = \sin\,\theta = \dfrac{\sqrt{2}}{\sqrt{29}}$

Answer: $\dfrac{\sqrt{2}}{\sqrt{29}}$

NOTE:

$$\cos\left(Arc\tan\frac{\sqrt{6}}{9}\right) = \cos\theta = \frac{9}{\sqrt{87}}$$

$$\sec\left(Arc\tan\frac{\sqrt{6}}{9}\right) = \sec\theta = \frac{\sqrt{87}}{9}$$

$$\csc\left(Arc\tan\frac{\sqrt{6}}{9}\right) = \csc\theta = \frac{\sqrt{29}}{\sqrt{2}}$$

$$\tan\left(Arc\tan\frac{\sqrt{6}}{9}\right) = \tan\theta = \frac{\sqrt{6}}{9}$$

$$\cot\left(\mathit{Arc}\tan\frac{\sqrt{6}}{9}\right) = \cot\theta = \frac{9}{\sqrt{6}}$$

Create your own example: If the angle is $\mathit{Arc}\tan\frac{a}{b}$, where the numbers a and b are positive, then

$$\cos\left(\mathit{Arc}\tan\frac{a}{b}\right) = \frac{b}{\sqrt{a^2 + b^2}}$$

$$\sin\left(\mathit{Arc}\tan\frac{a}{b}\right) = \frac{a}{\sqrt{a^2 + b^2}}$$

$$\tan\left(\mathit{Arc}\tan\frac{a}{b}\right) = \frac{a}{b}$$

$$\sec\left(\mathit{Arc}\tan\frac{a}{b}\right) = \frac{\sqrt{a^2 + b^2}}{b}$$

$$\csc\left(\mathit{Arc}\tan\frac{a}{b}\right) = \frac{\sqrt{a^2 + b^2}}{a}$$

$$\cot\left(\mathit{Arc}\tan\frac{a}{b}\right) = \frac{b}{a}$$

NOTE: Since a and b are positive numbers, then the number $\frac{a}{b}$ is **positive**. Thus, the angle $\mathit{Arc}\tan\frac{a}{b}$ is in the **I quadrant**.

Back to Topics List