MAXIMAL SUBRINGS AND E-GROUPS

C.J. Maxson

M.R. Pettet

Abstract

For a finite group G, let $E(G)$ denote the near-ring of functions generated by the semigroup, End(G), of endomorphisms of G. We characterize when $E(G)$ is maximal as a subnear-ring of $M_0(G)$. A group G is an E-group if $E(G)$ is a ring. We give a new characterization of finite E-groups and investigate the problem of determining, for finite E-groups, when $E(G)$ is maximal as a ring in $M_0(G)$.

2000 Mathematics Subject Classification. Primary 16Y30; Secondary 20E99, 20F99.

Key Words. Endomorphism near-rings; E-groups; Covers of groups.

I INTRODUCTION

Let $G = (G, +)$ be a group written additively, $+$, and with identity 0 and let $M_0(G) := \{f\colon\ G \to G | f(0) = 0\}$ be the near-ring of zero preserving functions on G. Recently ([12]), Neumaier characterized all maximal subnear-rings of $M_0(G)$ for finite G. Now, when G is abelian, the collection $\mathrm{End}(G)$ of endomorphisms of G is a ring in $M_0(G)$. Using results of Neumaier, Kreuzer and Maxson, ([8]) showed that for any abelian group G, $\mathrm{End}(G)$ is a maximal subnear-ring of $M_0(G)$ if and only if $G \cong \mathbb{Z}_3$ or G is a finite elementary abelian 2-group, i.e., $G \cong (\mathbb{Z}_2)^n$ for some positive integer n. It was also established in [8] that, for any torsion abelian group G, $\mathrm{End}(G)$ is a maximal ring in $M_0(G)$.

As is well-known, for nonabelian groups G, the sum of two endomorphisms need not be an endomorphism. Thus for nonabelian groups G, one considers the subnear-ring $E(G)$ of $M_0(G)$, generated by the semigroup, $\mathrm{End}(G)$ of endomorphisms of G. When G is finite each element of $E(G)$ is a sum of endomorphisms and we use this in the sequel without further mention.

In this paper we consider the natural extensions of [8] to finite nonabelian groups. In particular, for a finite nonabelian group G, we consider the following two questions:

(Q1) When is $E(G)$ maximal as a subnear-ring of $M_0(G)$?

(Q2) If $E(G)$ is a ring, when is $E(G)$ maximal as a ring in $M_0(G)$?

In the next section we give a complete answer to (Q1). We also find a rather curious connection with the action of the automorphism group, $\mathrm{Aut}(G)$, on G.

To investigate (Q2) one should know when $E(G)$ is a ring. This happens when G is an E-group. A short history, some known results and some new results on E-groups are given in Section III. In particular, we give a new characterization of E-groups similar to that known for I-groups. (Recall a group G is an I-group if the near-ring, $I(G)$, generated by the inner automorphisms of G is a ring.)

Then in Section IV we give some partial results on (Q2). However, a characterization of those finite E-groups G for which $E(G)$ is maximal as a ring in $M_0(G)$ remains open.

Convention: For the remainder of this paper, unless stated to the contrary, G will denote a finite group written additively. Moreover, a "maximal substructure" always means a *proper* substructure. Recall that for $a, b \in G$, the commutator, $[a, b]$, is defined by $[a, b] = -a - b + a + b$. Moreover, for $x \in G$, $\alpha \in \mathrm{Aut}(G)$, $[x, \alpha] = -x + \alpha(x)$.

II $E(G)$ MAXIMAL AS A SUBNEAR-RING OF $M_0(G)$

In this short section we show that for any finite nonabelian group $G, E(G)$ is not a maximal subnear-ring of $M_0(G)$. Combining this with a result from [8] we get a characterization of those finite groups G for which $E(G)$ is a maximal subnear-ring of $M_0(G)$.

Lemma II.1. *Let G be a finite nonabelian group. G has no nonzero, proper fully invariant subgroups if and only if $E(G) = M_0(G)$.*

Proof. If G has a fully invariant subgroup H, $\{0\} \subsetneqq H \subsetneqq G$ then for each $\sigma \in E(G)$, $\sigma(H) \subseteq H$ but this is not true for all $f \in M_0(G)$. The converse is well-known, e.g. Meldrum's book, [11], Theorem 10.11. □

Theorem II.2. *For a finite nonabelian group $G, E(G)$ is not a maximal subnear-ring of $M_0(G)$.*

Proof. If $E(G)$ is a maximal subnear-ring of $M_0(G)$, then $E(G) \subsetneqq M_0(G)$ so, by the above lemma, G must have a proper, nonzero fully invariant subgroup, say I. Now I is a normal subgroup of G and $E(G)(I) \leq I$ so I is an $E(G)$-ideal of G. But then G is not a simple $E(G)$-module which contradicts Theorem 4.3 of [12]. Hence the result. □

Corollary II.3. *Let G be a finite group. Then $E(G)$ is a maximal subnear-ring of $M_0(G)$ if and only if $G \cong \mathbb{Z}_3$ or $G \cong (\mathbb{Z}_2)^n$, n a positive integer.*

Proof. The result follows directly from the above theorem and Theorem 2.3 of [8]. □

The groups of the above corollary have a further interesting characterization. Recall that a transitive permutation group, A, on some set X is *primitive* or *acts primitively* on X if the one point stabilizers A_x, $x \in X$, are maximal in A. If A is 2-transitive on X, it is straightforward to verify that A is primitive on X.

Now let $A = \mathrm{Aut}(G), G$ finite. If A is transitive on $G^* = G - \{0\}$, then all elements of G^* have the same prime order p. Now, since G is a p-group, the commutator subgroup, $[G, G]$ is a proper subgroup, and, using the transitivity of A we find $[G, G] = \{0\}$. Thus G is an elementary abelian p-group, a vector space over $\mathbb{Z}_p$. For $x \in G^*$, $A_x \leq N_A(\langle x \rangle) \leq A$ and since $N_A(\langle x \rangle)$ contains all the scalar multiplications, $|N_A(\langle x \rangle)/A_x| = |\mathrm{Aut}(\langle x \rangle)| = p - 1$. If $N_A(\langle x \rangle) = A_x$ then $p - 1 = 1$ or $p = 2$.

Thus, if $p > 2$ then $N_A(\langle x \rangle) = A$ and so $\dim_{\mathbb{Z}_p}(G) = 1$. Then $|A/A_x| = |N_a(\langle x \rangle)/A_x| = p - 1$. Since A_x is maximal in $A, p - 1$ must be a prime so $p = 3$.

Conversely if $G \cong \mathbb{Z}_3$ or $G \cong (\mathbb{Z}_2)^n$, $n \geq 1$, then $A = \mathrm{Aut}(G)$ acts primitively on G^*. For if $C \cong \mathbb{Z}_3$ or $G \cong \mathbb{Z}_2$, this is clear. Furthermore, if $G \cong (\mathbb{Z}_2)^n$, $n \geq 2$, then any pair of distinct nonzero elements of G can be extended to an ordered basis and so we see $\mathrm{Aut}(G)$ is 2-transitive on G^*. From above, we get that $\mathrm{Aut}(G)$ is primitive on G^*.

Corollary II.4. *Let G be a finite group. The following are equivalent:*

i) *$E(G)$ is a maximal subnear-ring of $M_0(G)$;*

ii) *$G \cong \mathbb{Z}_3$ or $G \cong (\mathbb{Z}_2)^n, n \geq 1$;*

iii) *$\mathrm{Aut}(G)$ is primitive on G^*.*

It would be interesting and hopefully instructive to have a direct proof of the equivalence of i) and iii) in the above corollary. That is, how does the maximality of $E(G)$ follow from the maximality of the one-point stabilizers $\mathrm{Aut}(G)_x$, $x \in G^*$?

III *E*-GROUPS

Recall that an E-group is a group G such that the subnear-ring, $E(G)$, generated by the semigroup, $\mathrm{End}(G)$, of endomorphisms of G is a ring. Every abelian group is an E-group and these were the only known examples until the early 1970's. The first examples of nonabelian E-groups were given by Faudree, [5], in 1971. For more details on the history and some preliminary results, see Section 3 of Malone's expository paper, [10].

Prior to the discovery of these nonabelian E-groups, A. Chandy [4] characterized I-groups, i.e. those groups such that the near-rings, $I(G)$, generated by the inner automorphisms of G is a ring. These are 2-Engel groups which, as is known, can be characterized by the property that the centralizer, $C_G(x)$, of each element $x \in G$ is a normal subgroup of G [15]. Of course one also has A-groups, i.e., those groups G such that the near-ring, $A(G)$, generated by the group of automorphisms, $\mathrm{Aut}(G)$ is a ring. For further details on I-groups, A-groups, and E-groups we again refer to Malone's paper, [10], and to the exposition by Saad and Thomsen, [16]. In their paper, Saad and Thomsen mention that no characterization of E-groups like that of I-groups is known, nor are there any examples of A-groups that are not E-groups. In this section we give such a characterization of E-groups and also give an example of an A-group which is not an E-group.

Suppose G is a finite nilpotent group so G is the direct sum of its Sylow p-subgroups, S_p. Since the restriction of each $\sigma \in E(G)$ to S_p determines a map in $E(S_p)$, one finds that $E(G) \cong \oplus E(S_p)$.

From this we see that, to investigate finite E-groups, it suffices to consider p-groups that are also E-groups. We often refer to such groups as pE-groups.

Our first result in this section is essentially Theorem 3.1 of Caranti, [3], but without the hypothesis of nilpotence class 2.

Lemma III.1 (Caranti, [3])**.** *Let G be a finite p-group, $p > 2$. Then following are equivalent:*

i) *G is an E-group;*

ii) $[a, \alpha(a)] = 0,\ \forall a \in G,\ \forall \alpha \in \mathrm{End}(G)$*;*

iii) $[\alpha(a), b] = [a, \alpha(b)],\ \forall a, b \in G,\ \forall \alpha \in \mathrm{End}(G)$*;*

iv) $[\alpha(a), \beta(b)] = [\beta(a), \alpha(b)],\ \forall a, b \in G,\ \forall \alpha, \beta \in \mathrm{End}(G)$.

Proof. If G is an E-group, then $E(G)$ has abelian addition. Hence each $\alpha \in \mathrm{End}(G)$, commutes with the identity G. Thus i) $\Rightarrow$ ii). We conclude by showing ii) $\Rightarrow$ iii) since the remainder is as in Caranti, [3]. Let $a, b \in G$. Then, by ii), $a - b$ commutes with $\alpha(a - b) = \alpha(a) - \alpha(b)$, so

$$\alpha(a) - \alpha(b) + a - b = a - b + \alpha(a) - \alpha(b).$$

Adding $-a$ to the left and b to the right on both sides of this equation, we obtain

$$-a + \alpha(a) - \alpha(b) + a = -b + \alpha(a) - \alpha(b) + b.$$

Because $-a$ and b commute with $\alpha(a)$ and $-\alpha(b)$ respectively,

$$\alpha(a) - a - \alpha(b) + a = -b + \alpha(a) + b - \alpha(b).$$

Adding $-\alpha(a)$ to the left and $\alpha(b)$ to the right on both sides of this equation yields

$$-a - \alpha(b) + a + \alpha(b) = -\alpha(a) - b + \alpha(a) + b$$

so $[a, \alpha(b)] = [\alpha(a), b]$. □

We note that Jabara ([7], Lemma 1a) also realized that part iii) of the above lemma holds without the nilpotence class 2 hypothesis.

Corollary III.2. *If G is a finite E-group of odd order, then the center of $G, Z(G)$, is a fully invariant subgroup of G.*

Proof. Let $c \in Z(G)$ and $\alpha \in \mathrm{End}(G)$. Then from iii) of the above lemma, $[\alpha(c), a] = [c, \alpha(a)] = 0$ since $c \in Z(G)$. Thus $\alpha(c) \in Z(G)$. $\square$

The results in Sections 3 and 4 of [3] depend on the above lemma. A study of these sections show that the results of the lemma are used but not the nilpotence hypothesis of class 2. Thus this hypothesis can be omitted in these results. As an example we now give a (corrected) new proof of Caranti's Theorem 3.6 without the class 2 or exponent p^2 hypotheses.

As usual, we let $\mathrm{Aut}_c(G) := \{\alpha \in \mathrm{Aut}(G) | -g + \alpha(g) \in Z(G),\ \forall g \in G\}$ denote the normal subgroup of central automorphisms of G.

Theorem III.3 ([3], Theorem 3.6)**.** *If G is a pE-group of odd order, then* $\mathrm{Aut}(G)/\mathrm{Aut}_c(G)$ *is abelian of odd order.*

Proof. Let $\alpha, \beta \in \mathrm{Aut}(G)$. Three applications of part iii) of Lemma III.1 give for $x, y \in G$,

$$[x, [\alpha, \beta]y] = [x, (\beta\alpha)^{-1}\alpha\beta(y)] = [\alpha^{-1}\beta^{-1}(x), \alpha\beta(y)] = [\beta^{-1}(x), \beta(y)] = [x, y]$$

so $x^{[\alpha,\beta](y)} = x^y$. Therefore $[\alpha, \beta](y) - y \in Z(G)$ for all $y \in G$. Thus $[\alpha, \beta] \in \mathrm{Aut}_c(G)$. But this means $[\mathrm{Aut}(G), \mathrm{Aut}(G)] \leq \mathrm{Aut}_c(G)$ so $\mathrm{Aut}(G)/\mathrm{Aut}_c(G)$ is abelian. (This is essentially the proof given by Jabara [7], Proposition 3.)

For the second part, suppose $\alpha\ \mathrm{Aut}_c(G)$ is an element of order 2 in $\mathrm{Aut}(G)/\mathrm{Aut}_c(G)$ so $\alpha^2 \in \mathrm{Aut}_c(G)$. For any $x, y \in G$, again using (iii) of the above lemma, we get $\alpha[x, y] = [\alpha(x), \alpha(y)] = [\alpha^2(x), y] = [x + [x, \alpha^2], y] = [x, y]$ since $[x, \alpha^2] \in Z(G)$. (Recall $[x, \alpha^2] = -x + \alpha^2(x)$ and we have shown above that $\alpha^2 \in \mathrm{Aut}_c(G)$.) Therefore $[x, y] \in C_G(\alpha) =: \{g \in G | \alpha(g) = g\}$, i.e., $[G, G] \leq C(\alpha)$. If $e = \exp G/[G, G]$ then $e[G, \alpha] \leq eG \leq [G, G] \leq C_G(\alpha)$ and so, for any $x \in G$, $e[x, \alpha] = \alpha(e[x, \alpha]) = e\alpha[x, \alpha] = e(-[x, \alpha] + [x, \alpha^2]) = -e([x, \alpha]) + e[x, \alpha^2]$ since $[x, \alpha^2] \in Z(G)$. Therefore $e(-2[x, \alpha] + [x, \alpha^2]) = -2e[x, \alpha] + e[x, \alpha^2] = 0$. From Theorem 1 of [9], $-2[x, \alpha] + [x, \alpha^2] \in Z(G)$ which in turn gives $-2[x, \alpha] \in Z(G)$. Since G has odd order, $[x, \alpha] \in Z(G)$ which means $[G, \alpha] \leq Z(G)$, i.e., $\alpha \in \mathrm{Aut}_c(G)$, a contradiction. Therefore $\mathrm{Aut}(G)/\mathrm{Aut}_c(G)$ must be of odd order. $\square$

Corollary III.4. *If G is an E-group of odd order, then for every $x \in G$, $C_G(x)$ is a characteristic subgroup of G.*

Proof. For $\eta \in \mathrm{End}(G)$, $x \in G$, $y \in C_G(x)$, $[x, \eta^2(y)] = [\eta(x), \eta(y)] = \eta[x, y] = 0$, hence $\eta^2(g) \in C_G(x)$. This shows that $C_G(x)$ is invariant under η^2 for each $\eta \in \mathrm{End}(G)$. Now let $\alpha \in \mathrm{Aut}(G)$ and note that α^2 leaves $C_G(x)$ invariant. From the above theorem, there exists an odd integer m, say $m = 2k+1$, such that $\alpha^m \in \mathrm{Aut}_c(G)$. Now $C_G(x)$ is also invariant under α^m. In fact, if $y \in C_G(x)$, then $\alpha^m(y) = y + [y, \alpha^m] \in C_G(x)$ since $[y, \alpha^m] \in Z(G)$. Therefore $\alpha = \alpha^{m-2k} = \alpha^m(\alpha^{-k})^2$ leaves $C_G(x)$ invariant, as required. □

As mentioned above, I-groups are characterized as those groups G such that $C_G(x) \trianglelefteq G$, for each x in G, and that no similar characterization of E-groups is known, [10], [16]. Using the above results and results from Sections 3 and 4 of Caranti, [3], valid without the nilpotence class 2 hypothesis, we are now able to give this desired characterization.

Theorem III.5. *Let G be a finite p-group, $p > 2$. Then G is an E-group if and only if, for each $x \in G$, $C_G(x)$ is a fully invariant subgroup.*

Proof. Clearly if $C_G(x)$ is fully invariant then for each $x \in \mathrm{End}(G)$, $\alpha(x) \in C_G(x)$ so $[x, \alpha(x)] = 0$. From Lemma III.1, G is an E-group. Conversely, let $G = H_1 \oplus \cdots \oplus H_n$ where each H_i is indecomposable and let $x \in G$, say $x = h_1 + \cdots + h_n$, $h_i \in H_i$. Then $C_G(x) = \bigcap_{i=1}^{n} C_G(h_i)$ and so it suffices to consider the case $x = h_1 \in H_1$. In this case $C_G(x) = C_{H_1}(x) \oplus \cdots \oplus H_n$. Any automorphism of H_i can be extended to an endomorphism of G by defining it to be the zero map on H_j, $j \neq i$ and since, by the previous corollary, $C_{H_1}(x)$ is characteristic in H_1, we have that $C_G(x)$ is invariant under all such endomorphisms of G. Also any $\eta \in \mathrm{Hom}(G, Z(G))$ leaves $C_G(x)$ invariant. Now from [3] Theorem 4.3, every endomorphism α of G has the form $\alpha = \alpha_1 + \cdots + \alpha_n + \eta$ where $\alpha_i \in \mathrm{Aut}(H_i) \cup \{0\}$ and $\eta \in \mathrm{Hom}(G, Z(G))$ and so, $\alpha(C_G(x)) \subseteq C_G(x)$ for each $x \in G$. □

One would now conjecture that the "natural" characterization of A-groups would be "If G is a finite p-group, $p > 2$ then G is an A-group if and only if for each $x \in G$, $C_G(x)$ is a characteristic subgroup of G".

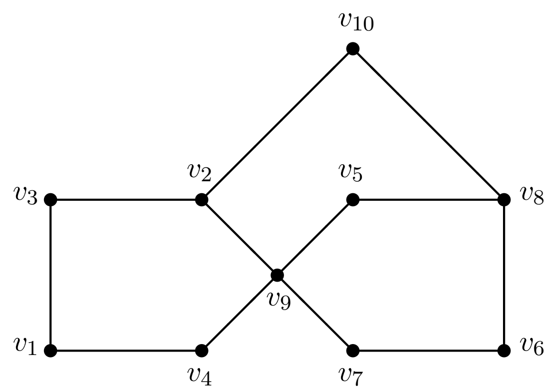
The proof of Theorem III.5 uses a result of Malone, [9] which depends on the construction of a certain endomorphism which need not be an automorphism and so cannot be used directly in a characterization of A-groups. Thus a characterization remains open at this time. However we can answer another query of Saad and Thomsen, [16].

Theorem III.6. *For any prime p, there exists a finite p-group P of nilpotence class 2 and exponent p^2 such that $\mathrm{Aut}(P) = \mathrm{Aut}_c(P)$ (and, in particular, P is an A-group) but P is not an E-group.*

Proof. The construction, employing a graph to define a presentation, is similar to one first introduced in [6] and subsequently adapted with modifications by several authors. The version used in

[14] seems most convenient for our purposes.

Let Γ be the (undirected) graph shown below with vertex set $V(\Gamma) = \{v_1, v_2, \ldots, v_{10}\}$ and edge set $E(\Gamma)$.



Let F be the free group of rank 10 on $V(\Gamma)$ and define an endomorphism θ of F by

$$\theta(v_1) = v_3, \quad \theta(v_2) = v_4, \quad \theta(v_5) = v_7, \quad \theta(v_6) = v_8, \quad \theta(v_9) = 0,$$
$$\theta(v_3) = v_1, \quad \theta(v_4) = v_2, \quad \theta(v_7) = v_5, \quad \theta(v_8) = v_6, \quad \theta(v_{10}) = 0.$$

Let π be the permutation (1 2) (3 4) (5 6) (7 8) (9 10) and for any prime p, let

$$R_p = \{-pv_i + [v_i, v_{\pi(i)}], [v_r, v_s], [v_i, v_j, v_k] |\ 1 \leq i, j, k \leq 10, \{v_r, v_s\} \in E(\Gamma)\} \subset F.$$

$R_p \cup \{0\}$ is invariant under θ and so θ induces an endomorphism (which, by a harmless abuse of notation, we also denote by θ) of the quotient group $P = F/R_p^F$ (where R_p^F denotes the normal closure of R_p in F).

Let $x_i = v_i + R_p^F$ for $1 \leq i \leq 10$. Then $P = \langle x_1, x_2, \ldots, x_{10}\rangle$ is a finite p-group of class 2 and exponent p^2 satisfying the additional relations

$$px_1 = [x_1, x_2] = -px_2, \quad px_3 = [x_3, x_4] = -px_4, \qquad px_5 = [x_5, x_6] = -px_6,$$
$$px_7 = [x_7, x_8] = -px_8, \quad px_9 = [x_9, x_{10}] = -px_{10}, \quad [x_r, x_s] = 0 \text{ if } \{v_r, v_s\} \in E(\Gamma).$$

Note that in Γ, all vertices have degree at least 2, there are no cycles of length less than 5 and the automorphism group of Γ is trivial. Thus, Γ satisfies the conclusions of [14] Theorem 1 (with $G = 1$). Also, the permutation of $V(\Gamma)$ induced by π moves each vertex of Γ outside its closed neighborhood (i.e. to a different and non-adjacent vertex). It follows by the argument of Theorem 2 of [14] that $\mathrm{Aut}(P) = \mathrm{Aut}_c(P)$. In particular, P is an A-group since, if $x \in P$ and $\alpha \in \mathrm{Aut}(P)$, then $[\alpha(x), x] = [x + z, x] = 1$ (where $z = [x, \alpha] \in Z(P)$). However, P is not an E-group since $[x_2, \theta(x_2)] = [x_2, x_4] \neq 1$. $\square$

We conclude this section with a result rather unrelated to the above, but motivated by the recent activity on groups with special covers (e.g., [1], [2]). Recall that a cover C of a group G is a collection $\{H_\alpha\}$ of proper subgroups of G such that $G = \cup H_\alpha$. The subgroups H_α are called the cells of the cover. In this result we do not require G to be a *finite* group.

Theorem III.7. *A nonabelian group has a finite covering consisting of fully invariant abelian subgroups if and only if G is central by finite and G is an E-group.*

Proof. From a result of Baer, (see [13]), if G has a finite cover of abelian subgroups, then G is central by finite. Now let $C = \{C_\alpha\}$ be a cover by fully invariant subgroups and define $\mathcal{R}(C) := \{f \in M_0(G) | f|_{C_\alpha} \in \mathrm{End}(C_\alpha)\}$. Since the C_α are abelian, computations show that $\mathcal{R}(C)$ is a ring contained in $M_0(G)$. Since the C_α are fully invariant, $\mathrm{End}(G) \subseteq \mathcal{R}(C)$, so $E(G) \subseteq \mathcal{R}(C)$ and thus $E(G)$ is a ring, i.e. G is an E-group.

Conversely, if G is an E-group, $E(G)x = \{\sigma(x) | \sigma \in E(G)\}$ is a fully invariant abelian subgroup, for each $x \in G$. We note that the proof of ii) $\Rightarrow$ (iii) in Lemma III.1 does not require G to be finite so we also have, from Corollary III.2 that $Z(G)$ is a fully invariant subgroup of G. Hence $F_x := (E(G)x) + Z(G)$ is a fully invariant abelian subgroup for each $x \in G$. Since G is nonabelian, each F_x is a proper subgroup. Now, following [2], since G is central by finite, we let $T := \{x_1, \ldots, x_n\}$ be a transversal of $Z(G)$ in G. Then $G = \bigcup_{i=1}^{n} ((E(G)x_i) + Z(G))$ since each $y \in G$ can be written as some $x_i + w$, $w \in Z(G)$ and some $x_i \in T$. Consequently G has a finite cover by fully invariant abelian subgroups. $\square$

IV $E(G)$ MAXIMAL AS A RING IN $M_0(G)$

We now turn to the question (Q2) of the introduction, that is, if G is an E-group, when is $E(G)$ maximal as a ring in $M_0(G)$? We are able only to give a partial answer to this question.

Definition IV.1. A cover $C = \{H_i\}_{i=1}^n$ by subgroups H_i is *bad* if for all i, $\langle H_i \cap H_j | i \neq j\rangle = H_i$. Otherwise the cover is *good*.

Theorem IV.2. *Let G be a finite nonabelian pE-group, $p > 2$, of nilpotence class 2. If $E(G)$ is maximal as a ring in $M_0(G)$, then every cover of G by fully invariant subgroups is bad.*

Proof. Let G satisfy the hypotheses and let G^* denote the abelian group with the same underlying set as G but with group operation $*$ defined by $x * y = x + y + [y,x]^{1/2} = x^{\frac{1}{2}} + y + x^{\frac{1}{2}}$. (This is well-defined since G has odd order so the map $G \to G$, $x \mapsto x^2$ is a bijection.) Note that every subgroup of G is a subgroup of G^* and every endomorphism of G is an endomorphism of G^*, so $\mathrm{End}(G) \subseteq \mathrm{End}(G^*)$ (as sets).

Because G is an E-group, if $x \in G$ and $e_1, e_2 \in E(G)$ then $[e_1(x), e_2(x)] = 1$ and so $e_1(x)*e_2(x) = e_1(x) + e_2(x)$. Therefore, addition is defined unambiguously in $E(G)$, whether we regard it as a subset of $M_0(G)$ or as a subset of $M_0(G^*)$. Since G^* is abelian, it follows that $E(G)$ is a subring of $\mathrm{End}(G^*)$.

Suppose that $\{H_i\colon\ 1 \le i \le n\}$ is a good fully invariant covering of G and assume that $K_1 = \langle H_1 \cap H_j\colon\ j \neq 1\rangle \lneqq H_1$. For each i, let $Z_i = H_i \cap Z(G)$. Then Z_i is a non-trivial subgroup of $Z(H_i)$ that is fully invariant in G by III.2. If S is the set of all maps $g \in M_0(G)$ whose restriction to each H_i is in $\mathrm{Hom}(H_i, Z_i)$, S is a ring and $e+s = s+e$ for all $e \in E(G)$. In fact, if $R = E(G)+S$, R is a ring and S is an ideal in R.

Let $\eta \in \mathrm{Hom}(H_1/K_1, Z_1)$. Because $H_1 \cap H_j \le K_1$ for all $j \neq 1$, we may construct a map f on G by defining $f(x) = \eta(x + K_1)$ if $x \in H_1$ and $f(x) = 1$ if $x \notin H_1$. Then the maximality of $E(G)$ implies that $f \in E(G)$, hence, $f \in \mathrm{End}(G^*)$.

Because $G^* = H_1 \cup \langle H_j\colon\ j \neq 1\rangle$ and because no group can be the union of two proper subgroups, $G^* = \langle H_j\colon\ j \neq 1\rangle$. Since $f(x) = 1$ for all $x \notin H_1$, it follows that $f(x) = 1$ for all $x \in G^*$ and, in particular, $\eta(x) = 1$ for all $x \in H_1$. This proves that $\mathrm{Hom}(H_1/K_1, Z(H_1)) = 0$ and so $H_1 = K_1$, a contradiction.

This proves that all coverings of G by proper fully invariant subgroups are bad. $\square$

We now indicate an application of this theorem.

Suppose $\mathrm{End}(G) = \mathrm{Aut}(G) \cup \mathrm{Hom}(G, Z(G))$. If $\alpha \in \mathrm{Aut}(G)$ and $\eta \in \mathrm{Hom}(G, Z(G))$, then $\alpha + \eta \in \mathrm{End}(G) \backslash \mathrm{Hom}(G, Z(G))$, so $\alpha + \eta \in \mathrm{Aut}(G)$. In particular, $id + \eta \in \mathrm{Aut}(G)$ for all $\eta \in \mathrm{Hom}(G, Z(G))$ and so $\eta(x) = -x + (id + \eta)(x) = [x, id + \eta] \in [G,\ \mathrm{Aut}(G)]$ for all $x \in G$. Also, for $\alpha, \beta \in \mathrm{Aut}(G)$ and $x \in G$, $\beta[x, \alpha] = \beta(-x + \alpha(x)) = -\beta(x) + \beta\alpha(x) = -\beta(x) + x - x + \alpha(x) - \alpha(x) + \beta\alpha(x) = -(-x + \beta(x)) + (-x + \alpha(x)) + (-\alpha(x) + \beta(\alpha(x))) \in [G,\ \mathrm{Aut}(G)]$. Therefore $[G,\ \mathrm{Aut}(G)]$ is a fully invariant subgroup of G containing $[G,\ \mathrm{Inn}(G)] = [G, G]$. Moreover, each cyclic subgroup of $G/[G,\ \mathrm{Aut}(G)]$ is invariant under $\mathrm{End}(G)$. This follows from the observation that, if $x \in G$, then for $\eta \in \mathrm{Hom}(G, Z(G))$, $\eta(x) \in [G,\ \mathrm{Aut}(G)]$ while if $\alpha \in\ \mathrm{Aut}(G)$, then $\alpha(x) = x - x + \alpha(x)$.

Corollary IV.3. *Let G be a finite nonabelian pE-group, $p > 2$, of nilpotency class 2 such that* $\mathrm{End}(G) = \mathrm{Aut}(G) \cup \mathrm{Hom}(G, Z(G))$. *If $G/[G, \mathrm{Aut}(G)]$ is not cyclic then $E(G)$ is not maximal as a ring in $M_0(G)$.*

Proof. Let $\overline{G} := G/[G,\ \mathrm{Aut}(G)]$. Since this abelian group $\overline{G}$ is non-cyclic, so is

$$\overline{G}/p\overline{G} = G/[G,\ \mathrm{Aut}(G)]/p(G/[G,\ \mathrm{Aut}(G)]) \cong G/([G,\ \mathrm{Aut}(G)] + pG).$$

Each cyclic subgroup of $G/([G,\ \mathrm{Aut}(G)] + pG)$ is invariant under $\mathrm{End}(G)$ and so, by lifting each such subgroup to a subgroup of G we obtain a cover of G by proper fully invariant subgroups. Since the intersection of any two distinct cells of the cover is contained in $([G,\ \mathrm{Aut}(G)] + pG)$, a proper subgroup of each cell, the cover is good, and the result follows from the previous theorem. □

The early nonabelian examples of E-groups were pE-groups, $p > 2$, having (among others) the following two properties:

($*$) $\mathrm{End}(G) = \mathrm{Aut}(G) \cup \mathrm{Hom}(G, Z(G))$, and

($**$) $\mathrm{Aut}(G) = \mathrm{Aut}_c(G)$.

On the other hand if G is a finite nonabelian p-group satisfying ($*$) and ($**$), then G is an E-group. Moreover ($**$) implies that every inner automorphism is central so G is of nilpotency class 2.

Corollary IV.4. *If G is a finite nonabelian p-group, $p > 2$ satisfying* ($*$) *and* ($**$) *above then $E(G)$ is not maximal as a ring in $M_0(G)$.*

Proof. By hypothesis, $[G,\ \mathrm{Aut}(G)] \leq Z(G)$. If $G/[G,\ \mathrm{Aut}(G)]$ is cyclic, then so is

$$(G/[G,\ \mathrm{Aut}(G)])/(Z(G)/[G,\ \mathrm{Aut}(G)]) \cong G/Z(G)$$

which contradicts the fact that G is nonabelian. Thus $G/[G,\ \mathrm{Aut}(G)]$ is not cyclic and the result now follows from the previous corollary. □

In particular, we mention that the examples of Faudree, [5], and those examples in Section 2 of Caranti, [3], satisfy ($*$) and ($**$). It should be mentioned that, in [3], Caranti also gives examples of pE-groups not satisfying ($**$). In this paper he also shows that if an E-group G is not the direct sum of two nonabelian groups then ($*$) is satisfied.

We remark that the authors have no example of a nonabelian E-group, G, for which $E(G)$ is a maximal ring in $M_0(G)$. This leads to the question, for a finite E-group G, if $E(G)$ is a maximal ring in $M_0(G)$ must G be abelian?

References

[1] Brodie, M., "Uniquely covered groups", Alg. Coll., 10 (2003), 101–108.

[2] Brodie, M., Chamberlain, R. and Kappe, L.-C., "Finite coverings by normal subgroups", Proc. Amer. Math. Soc., 104 (1988), 669–674.

[3] Caranti, A., "Finite p-groups of exponent p^2 in which each element commutes with its endomorphic images", J. Alg., 97 (1985), 1–13.

[4] Chandy, A., "Rings generated by the inner automorphisms of nonabelian groups", Proc. Amer. Math. Soc., 30 (1971), 59–60.

[5] Faudree, R., "Groups in which each element commutes with its endomorphic images", Proc. Amer. Math. Soc., 27 (1971), 236–240.

[6] Heineken, H. and Liebeck, H., "The occurrence of finite groups in the automorphism groups of nilpotent groups of class 2", Arch. Math., 25 (1974), 8–16.

[7] Jabara, E., "Automorphisms that fix the centralizers of a group", Rend. Sem. Mat. Univ. Padova, 102 (1999), 233–239.

[8] Kreuzer, A. and Maxson, C.J., "E-locally cyclic abelian groups and maximal near-rings of mappings", Forum Math., 18 (2006), 107–114.

[9] Malone, J.J., "More on groups in which each element commutes with its endomorphic images", Proc. Amer. Math. Soc., 65 (1977), 209–214.

[10] Malone, J.J., "Endomorphism near-rings through the ages", Y. Fong et al. (eds.), Near-rings and Near-fields, Kluwer Acad. Pub., Dordrecht, 1995, 31–43.

[11] Meldrum, J.D.P., *Near-rings and their links with groups*, Research Notes in Mathematics, Vol. 134, Pitman Adv. Pub. Program, Boston, London, 1985.

[12] Neumaier, C., "The maximal subnear-rings of the near-ring of zero-preserving functions on a finite group", Comm. in Alg., 33 (2005), 2499–2518.

[13] Neumann, B.H., "Groups covered by finitely many cosets", Publ. Math. Debrecen, 3 (1954), 227–242.

[14] Pettet, M.R., "Characterizing inner automorphisms of groups", Arch. Math., 55 (1990), 422–428.

[15] Robinson, D.J.S., *A course in the theory of groups*, Springer-Verlag, Berlin-Heidelberg-New York, 1982.

[16] Saad, G. and Thomsen, M., "Endomorphism near-rings: Foundations, problems, and recent results", Discrete Math., 208/209 (1999), 507–527.

C.J. Maxson
Department of Mathematics
Texas A&M University
College Station, TX 77843
USA
and
Department of Mathematics
7600 Stellenbosch
SOUTH AFRICA
email: cjmaxson@math.tamu.edu

M.R. Pettet
Department of Mathematics
University of Toledo
Toledo, OH 43603
USA
e-mail: mpettet@math.utoledo.edu