THE AUTOMORPHISM GROUP OF A GRAPH PRODUCT OF GROUPS

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Abstract

A theorem of Karrass, Pietrowski and Solitar on the structure of the automorphism group of an amalgamated free product is extended to automorphism groups of fundamental groups of graphs of groups in which the edge groups are incomparable up to conjugacy.

1 Introduction

The primary motivation for this note is the following theorem of Karrass, Pietrowski and Solitar [1]: Let G be an amalgamated free product $H *_U K$, where $H \neq U \neq K$ and assume that U is *conjugate maximal* in each of H and K (i.e. no conjugate of U in either factor properly contains U). If A is the group of automorphisms of G which map each of the factors H and K to a conjugate of either itself or the other factor, then A also admits an amalgamated free product decomposition.

It seems natural to ask to what extent an analogous conclusion holds for products involving more than two factors or, more generally, for fundamental groups of graphs of groups (the "graph products" of the title). In this paper an extension of the Karrass-Pietrowski-Solitar result is obtained by replacing conjugate maximality with a hypothesis on the edge groups (the "EGI hypothesis") which is generally more restrictive but which, in the case of two factors (and also HNN extensions), is equivalent to it. In essence, we show that if G is the fundamental group of a graph of groups which satisfies this hypothesis and if A is the group of those automorphisms of G which map vertex groups to conjugates of vertex groups then the action of G on the associated standard tree extends to an action of A and hence, A admits a combinatorial decomposition resembling that of G.

The argument depends heavily on basic results from the Bass-Serre theory of group actions on trees. Most of our notation and terminology follows that of [2].

If G is a group, an *inverted* edge of a G-graph is one which is mapped to itself by some element of G but with the orientation reversed (i.e. with the incident vertices interchanged). Such an edge is *subdivided* by adding a new vertex at its mid-point and replacing it with two adjacent edges which are interchanged by the elements of G which inverted the original edge. (Subdivision of the inverted edges is thus a topologically harmless device for eliminating inversions from a G-graph.)

We consider a graph of groups $(G(-), D)$ where D is a connected *directed* graph and for $\mathbf{v} \in VD$ and $\mathbf{e} \in ED$, $G(\mathbf{v})$ and $G(\mathbf{e})$ denote the corresponding vertex and edge groups (which will be treated as subgroups of the fundamental group $G = \pi_1(G(-), D)$). $G(\mathbf{e})$ is taken to be a subgroup of $G(\mathbf{v})$ if $\mathbf{v}$ is the initial vertex of $\mathbf{e}$. The edge labels serve also as names of embeddings and as names of certain elements of the generating set of G. Thus, if $\mathbf{v}$ is the terminal vertex of $\mathbf{e}$, then $\mathbf{e}$ denotes both an embedding of $G(\mathbf{e})$ in $G(\mathbf{v})$ and also an element of G (possibly the identity) which conjugates elements of $G(\mathbf{e})$ to their images in $G(\mathbf{v})$ under the map $\mathbf{e}$. Of course, in the case that G is an amalgamated free product, D is simply a segment and in the case of an HNN extension, it is a loop.

Some additional *ad hoc* terminology will facilitate the statement of the theorem. If $\mathbf{v}$ is a vertex of D, the subgroups of $G(\mathbf{v})$ of the form $G(\mathbf{e})$ (where $\mathbf{e}$ is an outgoing edge) and $G(\mathbf{e})^{\mathbf{e}}$ (where $\mathbf{e}$ is an incoming edge) will both be called $\mathbf{v}$-*incident edge groups* corresponding to $\mathbf{e}$. We shall consider only graphs of groups which are *proper* in the sense that for each vertex $\mathbf{v}$ of G, every $\mathbf{v}$-incident edge group which does not correspond to a loop at $\mathbf{v}$ is a proper subgroup of $G(\mathbf{v})$. One class of proper graphs of groups which we will need to avoid is those in which D consists of a single loop with both edge groups equal to the full vertex group. These we shall refer to as *degenerate loops*. (If $(G(-), D)$ is a degenerate loop then G is simply a semidirect product of the vertex group $G(\mathbf{v})$ with an infinite cyclic group and $A = N_{AutG}(G(\mathbf{v}))$.)

The central hypothesis of the paper is the following:

Definition. We shall say that $(G(-), D)$ satisfies the *edge group incomparability (or EGI) hypothesis* if, for each vertex $\mathbf{v}$ of D, no $\mathbf{v}$-incident edge group is conjugate in $G(\mathbf{v})$ to a subgroup of another $\mathbf{v}$-incident edge group unless the two are identical and correspond to the same edge of D.

For example, the EGI hypothesis is obviously satisfied if the edge groups are all finite with none having order divisible by the order of another.

If Γ is any directed graph, we will denote the underlying undirected graph by Γ_0. Our main objective is

Theorem 1 *Let G be the fundamental group of a proper graph of groups $(G(-), D)$ which satisfies the EGI hypothesis and is not a degenerate loop. Let T be the corresponding standard G-tree and let A be the group of all automorphisms of G which map each vertex group of $(G(-), D)$ to a G-conjugate of some vertex group. Then T is an $InnG$-tree and the action of $InnG$ on the underlying undirected tree T_0 extends to an action of A which, in turn, induces an action of $A/InnG$ on the undirected graph D_0.*

As a consequence, A is isomorphic to the fundamental group of a proper graph of groups $(A(-), D^*)$ which satisfies the EGI hypothesis and such that the following hold:

(i) *D_0^* is isomorphic to the quotient modulo A of the A-graph obtained from D_0 by subdividing all inverted edges.*

(ii) *If $\bar{\mathbf{v}} \in VD_0^*$ is the image of a vertex $\mathbf{v}$ of D_0 then $A(\bar{\mathbf{v}}) = N_A(G(\mathbf{v}))$. If $\bar{\mathbf{v}} \in VD_0^*$ arises from subdivision of an edge of D_0 which is inverted under the action of A then $A(\bar{\mathbf{v}})$ consists of those elements of A which either fix or invert this edge.*

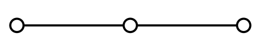
(iii) *$|ED^*| \leq |ED|$. Moreover, if D is a finite tree then so is D^*.*

The inspiration for the proof of the theorem is the observation that if $(G(-), D)$ is a proper graph of groups which satisfies the EGI hypothesis and is not a degenerate loop, then there is a G-bijection between the vertex set VT of the standard G-tree T and the A-set of all vertex stabilizers in G (Lemma 4.1). The bulk of the argument is concerned with establishing that under the EGI hypothesis, the action of A on VT induced by this bijection is actually isometric on the underlying undirected tree T_0 (Lemma 3.1).

As a simple application, let H and K be groups with proper non-trivial subgroups U and V respectively and consider the group

$$G = \langle H, K : [u, v] = 1 \ \forall u \in U, v \in V \rangle.$$

If $L = U \times V$, then $G = H *_U L *_V K$ and since U and V are normal in L, the EGI hypothesis holds. Here the graph D_0 is



If the non-identity automorphism of D_0 is realized by an element of A (the case if, for example, there is an isomorphism $\phi : H \to K$ such that $U^\phi = V$) then D_0^* is a segment and Theorem 1 yields that A is an amalgamated free product $N_A(H) *_B N_A(L)$, where $B = N_A(H) \cap N_A(L)$. Otherwise, $D_0^* \cong D_0$ and so A has the form $N_A(H) *_B N_A(L) *_C N_A(K)$.

Theorem 1 is probably of greatest interest in the case that the subgroup A is the full automorphism group of G. One additional hypothesis which ensures that $A = AutG$ is that each vertex group of $(G(-), D)$ possess intrinsically the property that it fixes a vertex of *any* tree on which it acts. (This is termed *property (FA)* in [4] where it is shown to be characterized by three conditions: that G have no non-trivial amalgamated free product decomposition, that it have no infinite cyclic quotient, and that it not be a union of a strictly ascending chain of proper subgroups.) In this case, if v is a vertex of the G-tree T and if $\alpha \in AutG$, then $(G_v)^\alpha \subseteq G_u$ for some $u \in VT$ and similarly, $(G_u)^{\alpha^{-1}} \subseteq G_w$ for some $w \in VT$, whence $G_v \subseteq G_w$. The EGI hypothesis and the fact that $(G(-), D)$ is proper then force $v = w$ (Lemma 4.1) and so $(G_v)^\alpha = G_u$, proving that α maps vertex stabilizers to vertex stabilizers and so belongs to A. It follows, for example, that $A = AutG$ if each vertex group of $(G(-), D)$ is finitely generated and periodic. (See 6.3.1 of [4].) Theorem 2.3 of [5] provides another criterion of this type which applies when the vertex groups are polycyclic-by-finite.

Following the completion of the proof of Theorem 1 in Section 4, we derive an analog of the Karrass-Pietrowski-Solitar theorem for proper HNN extensions $G = H *_U t$ (Theorem 2). This theme is pursued further in Section 5 where, in the special case that U is centralized by t, a short computation leads to a very precise description of A in terms of H and U. The final observation in Section 4 is a natural extension of a result of E. Raptis, D. Varsos and O. Talelli on automorphism groups of amalgamated free products and HNN extensions of polycyclic-by-finite groups (Theorem 3).

2 Preliminaries

Let G be a group acting on a set X. We shall say that the action is *locally transitive* if for every element x of X, $N_G(G_x)$ is transitive on the set $Fix(G_x)$ of points fixed by the stabilizer G_x of x or, equivalently, if for any two elements x and y of X, G_x is contained in G_y only if $G_x = G_y$ and the orbits x^G and y^G are identical. In the context of finite permutation groups, this property is a familiar (and trivial) consequence of transitivity (although in general, local transitivity does not follow from transitivity without the additional hypothesis that each point stabilizer be conjugate maximal).

From the Bass-Serre theory, the group G of Theorem 1 acts on the standard tree (or universal cover) T. Accordingly, most of the steps in the argument will be formulated in terms of group actions on trees. We say that the action of G on a graph Γ is *locally edge-transitive* if it is locally transitive on the edge set $E\Gamma$. If e and f are edges of the standard tree T with $G_e \subseteq G_f$ and if $e = e_1, e_2, \ldots, e_n = f$ is the edge sequence in the geodesic connecting e and f, then $G_e = G_e \cap G_f \subseteq G_{e_i}$ for $1 \leq i \leq n$. From this observation, the EGI hypothesis defined in the Introduction is easily seen to be simply a reformulation in terms of the graph of groups of the assumption that G acts locally edge-transitively on T.

The following observation is crucial here: If e is an edge of the G-tree T and if p and q are any two vertices of $Fix_T(G_e)$ then G_e fixes each vertex and edge in the unique reduced path (or *geodesic*) between p and q. (By G_e, we shall always mean the *point-wise* stabilizer of the edge e viewed as a 1-simplex.) In particular, G_e fixes each of the vertices incident with e.) It follows that $Fix_T(G_e)$ is a $N_G(G_e)$-tree and so (see, for example, §4 and §5 of [4]) if G acts locally edge-transitively and without inversion on T, then for each edge e of T with incident vertices u and v, $N_G(G_e)$ is either an amalgamated free product $N_{G_u}(G_e) *_{G_e} N_{G_v}(G_e)$ (if u and v are not $N_G(G_e)$-conjugate) or an HNN-extension $N_{G_v}(G_e) *_{G_e} t$ (if $u^t = v$ for some $t \in N_G(G_e)$). In particular, $N_G(G_e)$ is generated by those of its elements g for which e and e^g are adjacent.

Let T be a tree and let d be the edge metric on VT (so if u and v are vertices of T, $d(u,v)$ is the number of edges in the geodesic $[u,v]$ from u to v).

Lemma 2.1 *If G is a group and T is a simplicial G-tree, then any element of G which inverts an edge e neither fixes a vertex nor inverts any edge distinct from e.*

Proof. This is an easy consequence of the d-isometric action of G on T.

The thrust of the next lemma is that if G is a group of automorphisms of a tree T, then (even *without* the hypothesis of local edge-transitivity) elements of $N_{Sym(VT)}(G)$ map edge stabilizers to edge stabilizers.

Lemma 2.2 *Let T be a simplicial tree, $G \subseteq AutT$ and $\alpha \in N_{Sym(VT)}(G)$. Suppose that $e \in ET$ with incident vertices p and q, and let $p^\alpha = u$ and $q^\alpha = v$. Then $(G_e)^\alpha = G_f$ for some edge f in the geodesic $[u,v]$. Moreover, if $g \in G$ such that g inverts the edge e, then f may be chosen so that g^α inverts f.*

Proof. We have

$$(G_e)^\alpha = (G_p \cap G_q)^\alpha = G_u \cap G_v \subseteq G_b$$

for every edge b in the geodesic $[u,v]$. Since p and q lie in different connected components of $T\backslash\{e\}$, there is an edge f in $[u,v]$ with incident vertices x and y, say, such that α^{-1} maps x and y respectively to vertices z and w in different components of $T\backslash\{e\}$. Then e lies in the geodesic $[z,w]$ and so $G_z \cap G_w \subseteq G_e$. It follows that

$$G_f = G_x \cap G_y = (G_z \cap G_w)^\alpha \subseteq (G_e)^\alpha,$$

whence $(G_e)^\alpha = G_f$.

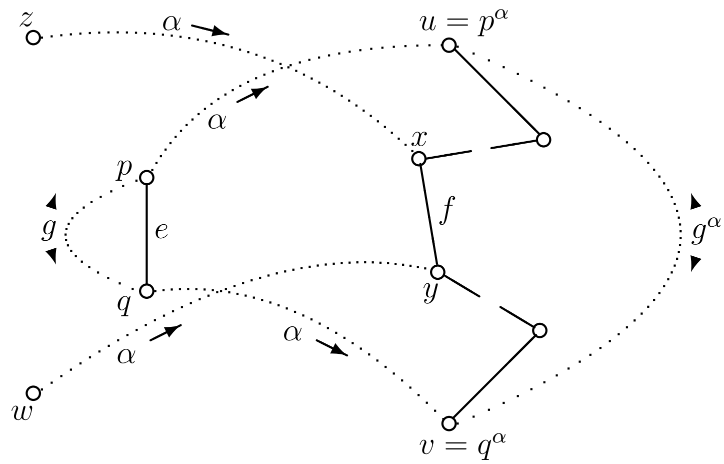
If g inverts the edge e (so $p^g = q$ and $q^g = p$), then

$$u^{g^\alpha} = p^{\alpha g^\alpha} = (p^g)^\alpha = q^\alpha = v.$$

By a similar argument, $v^{g^\alpha} = u$. By Lemma 2.1, g does not stabilize any vertex of T and so neither does g^α. Since g^α interchanges u and v, it follows that it must invert some edge f in $[u,v]$. Let x and y be the vertices incident with f (so g^α interchanges x and y) and let $x^{\alpha^{-1}} = z$ and $y^{\alpha^{-1}} = w$. Then

$$z^g = x^{\alpha^{-1}g} = x^{g^\alpha \alpha^{-1}} = y^{\alpha^{-1}} = w.$$

Similarly, $w^g = z$. Since g fixes no vertex of T, it must invert an edge in the geodesic $[z, w]$ and so by Lemma 2.1, e is an edge in $[z, w]$. Thus, z and w lie in different components of $T\backslash\{e\}$ and, as in the previous paragraph, $(G_e)^\alpha = G_f$.



Lemma 2.3 *Let G be a group and T be a G-tree. Let $\{u_1, u_2, \ldots u_n\}$ be a set of vertices of T and let p also be a vertex of T. If $g \in \langle G_{u_1}, G_{u_2}, \ldots G_{u_n}\rangle$, then $d(p, p^g) \equiv 0$ (mod 2).*

Proof. First, observe that if x, y, z are any three vertices of T, then

$$d(x, y) + d(y, z) \equiv d(x, z)\,(mod\,2).$$

We induct on the minimum length m of an expression of g as a product $g_1 g_2 \ldots g_m$ with consecutive terms belonging to distinct G_{u_j}'s. If $m = 1$, then $g \in G_{u_i}$ for some i and so

$$d(p, p^g) \equiv d(p, u_i) + d(u_i, p^g) \equiv 0\,(mod\,2)$$

because $d(u_i, p^g) = d((u_i)^g, p^g) = d(u_i, p)$. If $m > 1$, write $g = ha$, where $a \in G_{u_i}$ for some i and h has minimum length $m - 1$. If $q = p^h$, then $d(p, q) \equiv 0$ (mod 2) by the inductive hypothesis and so

$$d(p, p^g) \equiv d(p, q) + d(q, p^g) \equiv d(q, q^a) \equiv 0\,(mod\,2).$$

Lemma 2.4 *Let T be a simplicial G-tree with no inversions. If p and q are adjacent vertices of T sharing a common edge e, then $\langle G_p, G_q\rangle$ is an amalgamated free product $G_p *_{G_e} G_q$. If, in addition, $p^g = q$ for some $g \in G$, then $\langle G_p, g\rangle = \langle G_q, g\rangle$ is an HNN extension $G_q *_{G_e} g$.*

Proof. The set of all edges of T which are conjugate to e by an element of $\langle G_p, G_q\rangle$ is the edge set of a $\langle G_p, G_q\rangle$-subtree of T. Since $\langle G_p, G_q\rangle$ is edge transitive but (by Lemma 2.3) not vertex transitive on this subtree, the first conclusion follows from the Bass-Serre theory. If $p^g = q$, then the set of all edges of T which are conjugate to e by an element of $\langle G_p, g\rangle$ is a tree on which $\langle G_q, g\rangle$ is both edge and vertex transitive and again, the Bass-Serre theory yields the desired conclusion.

The next lemma (and its proof) is a slight variation on the key first step in [1].

Lemma 2.5 *Let $G = H *_U K$. If $g \in N_G(U)$ such that $\langle H, K^g\rangle = G$ and $H \cap K^g = U$, then $K^g = K^h$ for some $h \in N_H(U)$.*

Proof. This is trivial if $H = U$ or $K = U$. If $H \neq U \neq K$, let X and Y be right transversals for U in H and K respectively and assume that each contains 1. It is sufficient to prove that $g = kh$ for some $h \in H$ and $k \in K$ (for if so, $U^h = (H \cap K)^h = H \cap K^h = H \cap K^g = U$, whence $h \in N_H(U)$). If this is not so, then without loss of generality we may assume that for some $n \geq 1$, $g = x_1y_1x_2y_2 \ldots x_ny_n$ where $x_i \in X\backslash\{1\}$ and $y_i \in Y\backslash\{1\}$ for $1 \leq i \leq n$. But if $h_j \in X\backslash\{1\}$ and $k_j \in Y\backslash\{1\}$ for $1 \leq j \leq m$, then $h_1k_1^gh_2k_2^g \ldots h_mk_m^g$ is reduced and in particular, no such element belongs to K. It follows that $\langle H, K^g\rangle \cap K = U$ which contradicts the assumption that $\langle H, K^g\rangle = G$.

Finally in this section, we prove a result analogous to Lemma 2.5 for HNN extensions. Similar results (and proofs) appear in [3] and in [5].

Lemma 2.6 *Let G be an HNN extension $H *_U t$, where t normalizes the subgroup U. If $G = \langle H, s\rangle$ for some $s \in N_G(U)$, then $s = at^\varepsilon b$ for some $a, b \in H$ and $\varepsilon \in \{-1, 1\}$.*

Proof. Let X be a right transversal for U in H which contains the identity, whence s has a unique t-reduced representation $ht^{\varepsilon_1}x_1 \ldots t^{\varepsilon_n}x_n$ where $h \in H$ and where for $1 \leq i \leq n$, the x_i's are elements of X and the ε_i's are ± 1. The claim of the lemma is that $n = 1$ so assume otherwise, that $n > 1$. Clearly, it is no loss to assume that $h = x_n = 1$ and so $s = t^{\varepsilon_1}h_1 \ldots t^{\varepsilon_n}$. If N is the normal closure of H in G, then G/N is infinite cyclic with both tN and sN being generators and so $s \equiv t^{\pm 1}\,(mod\, N)$. Thus, $\sum_{i=1}^n \varepsilon_i = \pm 1$ and in particular, n is odd.

Computing a t-reduced expression for a positive power of s (or any element of G) involves eliminating t's and t^{-1}'s in pairs and hence, at most $(n-1)/2$ t-reductions can occur between consecutive occurrences of the expression for s in such a power. In other words, the power of t in the exact middle of each occurrence of the expression for s cannot be absorbed into any t-reductions. Since $n > 1$, it follows that the t-reduced expression for any non-trivial power of s has the form $ut^{\varepsilon_1}x_1wx_{n-1}t^{\varepsilon_n}$, where $u \in U$ and w involves at least one non-trivial power of t.

Because $G = \langle H, s\rangle$ and $s \in N_G(U)$, we may write $t = gs^{e_1}y_1 \ldots s^{e_m}y_m$ where $g \in H$, the y_i's are non-identity elements of X and the e_i's are non-zero integers for $1 \leq i \leq m$. From the conclusion obtained in the preceding paragraph about the reduced form of each non-trivial power of s, it follows that non-trivial powers of t must occur at least $3m$ times in the t-reduced form of $gs^{e_1}y_1 \dots s^{e_m}y_m$. This obviously contradicts the fact that this expression represents t itself.

3 Locally edge-transitive actions

A concisely formulated technical lemma represents the crux of the argument.

Lemma 3.1 *Let T be an undirected simplicial tree and suppose that G is a locally edge-transitive subgroup of $AutT$. Then $N_{Sym(VT)}(G) \subseteq AutT$.*

Proof. First we argue that it is sufficient to restrict our attention to the case that the action of G on T involves no edge inversions.

Let T^* be the G-tree obtained from T by subdividing each inverted edge (and *only* these edges). The action of G on T^* is still faithful and locally edge-transitive (since the set of edge-stabilizers is not changed) and there are, of course, no edge inversions. Let $A = N_{Sym(VT)}(G)$. With G identified as a subgroup of $AutT^*$, we must now show that A can be identified with a subgroup of $N_{Sym(VT^*)}(G)$.

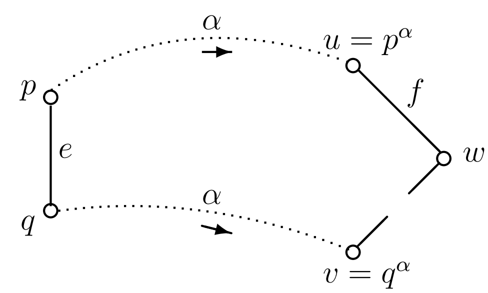
If w is the mid-point of an edge $e \in ET$ such that e is inverted by $g \in G$ (so $g \in N_G(G_e)$), then $G_w = G_e\langle g \rangle$. If $\alpha \in A$, Lemma 2.2 implies that there is an edge f such that $(G_e)^\alpha = G_f$ and g^α inverts f, whence

$$(G_w)^\alpha = (G_e)^\alpha \langle g^\alpha \rangle = G_f \langle g^\alpha \rangle = G_z,$$

where z is the midpoint of f. This vertex z of T^* is uniquely determined by w and α, for if $G_z = G_x$ with $x \in VT^*$, then g^α fixes x but fixes no vertex of T (Lemma 2.1), whence x is the mid-point of an edge b of T. Then g^α must either invert or fix b, which is only possible if $b = f$ and $x = z$ (again by Lemma 2.1). Thus, if w is the mid-point of an edge which is inverted by $g \in G$, we may define w^α to be the mid-point of the unique edge of T which is inverted by g^α. It is routine to check that this defines an action of A on the set of mid-points of inverted edges of T and so A may be identified as a subgroup of $N_{Sym(VT^*)}(G)$.

The upshot of this is that if we can show that $N_{Sym(VT^*)}(G) \subseteq AutT^*$, then $A \subseteq AutT^*$ and, because VT is A-invariant, it will follow that $A \subseteq AutT$ as required. Therefore, we may assume for the remainder of the proof that $T = T^*$ (that is, that G acts without inversion on T).

We must show that for any $\alpha \in A$ and any pair p, q of adjacent vertices of T, the images p^α and q^α are also adjacent. Suppose that p and q are such a pair of vertices sharing a common edge e. Let $u = p^\alpha$ and $v = q^\alpha$ and let f be the first edge in the geodesic $[u, v]$ and w be the first vertex out of u in this geodesic (so f is incident with u and w). Then $(G_e)^\alpha = (G_p \cap G_q)^\alpha = G_u \cap G_v$ is an edge stabilizer (Lemma 2.2) which fixes every edge in $[u, v]$. Therefore, local edge-transitivity yields that all edges in $[u, v]$ are conjugate to f under $N_G(G_f)$ (and hence, have stabilizer G_f). In particular, $(G_e)^\alpha = G_u \cap G_v = G_f$. We now consider two cases:



Case 1. p and q are not $N_G(G_e)$-conjugate.

In this case, u is certainly not $N_G(G_f)$-conjugate to v and so, since all edges in $[u, v]$ are $N_G(G_f)$-conjugate, w must be $N_G(G_f)$-conjugate to v (and not to u). Let g be an element of $N_G(G_f)$ such that $w^g = v$. Since u and w are adjacent but not conjugate in $N_G(G_f)$, local edge-transitivity implies that $N_G(G_f) = N_{G_u}(G_f) *_{G_f} N_{G_w}(G_f)$ and in particular, $g \in N_G(G_f) \subseteq \langle G_u, G_w\rangle$. Thus,

$$\langle G_u, G_v\rangle = \langle G_u, (G_w)^g\rangle \subseteq \langle G_u, G_w\rangle.$$

On the other hand, $g^{-1} = x^\alpha$ for some x in $N_G(G_e)$. Again by local edge-transitivity, $N_G(G_e) \subseteq \langle G_p, G_q\rangle$ and in particular, $x \in \langle G_p, G_q\rangle$. Thus,

$$\langle G_u, G_w\rangle = \langle G_p, (G_q)^x\rangle^\alpha \subseteq \langle G_p, G_q\rangle^\alpha = \langle G_u, G_v\rangle.$$

It follows from Lemma 2.4 that

$$G_u *_{G_f} G_w = \langle G_u, G_w\rangle = \langle G_u, G_v\rangle = \langle G_u, (G_w)^g\rangle.$$

Because

$$G_f = G_u \cap G_v = G_u \cap (G_w)^g,$$

Lemma 2.5 yields that $G_v = (G_w)^g = (G_w)^h$ for some $h \in N_{G_u}(G_f)$. Since w^h is adjacent to $u^h = u$, the geodesic $[w, v]$ is contained in $[w^h, v]$. Because each edge in $[u, v]$ (and hence, in $[w, v]$) has stabilizer G_f, we conclude that if $v \neq w$ (or $w^h \neq w$), then

$$G_v = (G_w)^h \cap G_v \subseteq G_f \subseteq G_u.$$

But if $G_v \subseteq G_u$, then

$$g \in \langle G_u, G_w\rangle = \langle G_u, G_v\rangle = G_u$$

whence, $d(u, v) = d(u^g, w^g) = d(u, w) = 1$ and so $v = w$. Thus, we are forced to the conclusion that $v = w$ in all cases, whence u and v are adjacent in T.

Case 2. p and q are $N_G(G_e)$-conjugate.

Let $p = q^t$ where $t \in N_G(G_e)$. Then $N_G(G_e) = N_{G_p}(G_e) *_{G_e} t$ and so $N_G(G_f) = N_G(G_e)^\alpha = N_{G_u}(G_f) *_{G_f} t^\alpha$. Also,

$$u^{t^\alpha} = (p^\alpha)^{t^\alpha} = (p^t)^\alpha = q^\alpha = v.$$

We claim first that u and w are $N_G(G_f)$-conjugate. For if not, then the action of $N_G(G_f)$ on $Fix(G_f)$ is not vertex-transitive and so

$$t^\alpha \in N_G(G_f) = N_{G_u}(G_f) *_{G_f} N_{G_w}(G_f) \subseteq \langle G_u, G_w \rangle.$$

Applying α^{-1}, we conclude that $t \in \langle G_p, G_r \rangle$, where $r^\alpha = w$, which contradicts Lemma 2.3 since $d(p, p^t) = d(p, q) = 1$.

Let $s \in N_G(G_f)$ such that $u = w^s$ (and so by Lemma 2.4, $N_G(G_f)$ is an HNN extension $N_{G_u}(G_f) *_{G_f} s$). Then $t^\alpha \in N_G(G_f) = \langle N_{G_u}(G_f), s \rangle$ and so $\langle G_u, t^\alpha \rangle \subseteq \langle G_u, s \rangle$. But

$$s^{\alpha^{-1}} \in N_G(G_f)^{\alpha^{-1}} = N_G(G_e) = N_{G_p}(G_e) *_{G_e} t \subseteq \langle G_p, t \rangle.$$

Therefore,

$$s \in \langle G_p, t \rangle^\alpha = \langle G_u, t^\alpha \rangle$$

and so $\langle G_u, s \rangle \subseteq \langle G_u, t^\alpha \rangle$. Hence, $\langle G_u, s \rangle = \langle G_u, t^\alpha \rangle$.

Since, by Lemma 2.4, $\langle G_u, s \rangle = G_u *_{G_f} s$, it follows from Lemma 2.6 that $t^\alpha = a s^\varepsilon b$, where $a, b \in G_u$ and $\varepsilon \in \{1, -1\}$. Therefore,

$$d(u, v) = d(u, u^{t^\alpha}) = d(u, u^{a s^\varepsilon b}) = d(u, u^{s^\varepsilon}).$$

If $\varepsilon = -1$, then

$$d(u, v) = d(u, u^{s^{-1}}) = d(u, w) = 1$$

and if $\varepsilon = 1$, then

$$d(u, v) = d(u, u^s) = d(u^{s^{-1}}, u) = d(w, u) = 1,$$

so in either case, u and v are adjacent. This completes the proof that α is actually a d-isometry of the (unoriented) tree T and hence, that $A \subseteq AutT$. Of course, in considering the implications of this conclusion for the structure of A, we must allow for the possibility that the action of A involves edge inversions, even if that of G is inversion-free.

The following observation about local transitivity is stated in somewhat greater generality than is strictly necessary for the proof of Theorem 1 but the price is only a short induction argument.

Lemma 3.2 *Let $A \subseteq Sym(X)$ for some set X. If A contains a locally transitive ascendant subgroup G, then A itself is locally transitive.*

Proof. Let $G = G_1 \trianglelefteq G_2 \trianglelefteq \ldots \trianglelefteq G_\beta = A$ be an ascending series. If the lemma is false, let δ be the smallest ordinal for which G_δ is not locally transitive. Let $K = G_\delta$.

Suppose δ is a limit ordinal. If x, $y \in X$ with $K_x \subseteq K_y$, then for every $\gamma < \delta$, G_γ is locally transitive and so $(G_\gamma)_x = (G_\gamma)_y$. Since K is the union of all such G_γ's, it follows that $K_x = K_y$, and this contradicts the assumption that K is not locally transitive.

Assume now that δ is not a limit ordinal and let $H = G_{\delta-1}$. If $x, y \in X$ with $K_x \subseteq K_y$, then $H_x \subseteq H_y$ and hence, by the local transitivity of H, $H_x = H_y$ and $y = x^g$ for some $g \in N_H(H_x)$ (so $x^K = y^K$). But $K_y \subseteq K_x H$, for if $\alpha \in K_y = (K_x)^g$ then

$$\alpha = \beta^g = \beta[g, \beta]^{-1} \in K_x H$$

for some $\beta \in K_x$. Therefore,

$$K_y = K_x H \cap K_y = K_x(H \cap K_y) = K_x.$$

This again contradicts the assumption that $K = G_\delta$ is not locally transitive.

Lemma 3.2 suggests the following slight refinement of Lemma 3.1 which follows from it by transfinite induction:

Corollary 3.3 *Let T be a simplicial tree and suppose that G is a locally edge-transitive subgroup of $AutT$. If $\alpha \in Sym(VT)$ such that G is an ascendant subgroup of $\langle G, \alpha \rangle$, then $\alpha \in AutT$.*

4 The main results

Lemma 4.1 *Let $(G(-), D)$ be a proper graph of groups which satisfies the EGI hypothesis and is not a degenerate loop. Let T be the corresponding standard G-tree. If $u, v \in VT$ such that $G_u \subseteq G_v$ then $u = v$.*

Proof. Let e be the first edge out of u in the geodesic from u to v and let w be the first vertex. If f is any edge incident with u then

$$G_f \subseteq G_u = G_u \cap G_v \subseteq G_e \subseteq G_w.$$

By the EGI hypothesis, f is G-conjugate to e and $G_f = G_e = G_u$. If $\mathbf{u}$ and $\mathbf{e}$ are the projections of u and e in D, then because $(G(-), D)$ is proper, $\mathbf{e}$ is a loop at $\mathbf{u}$ and $\mathbf{e}$ is the only edge of D which is incident with $\mathbf{u}$. Because D is connected, $\mathbf{e}$ must be the only edge in D and $(G(-), D)$ is a degenerate loop, a contradiction.

Proof of Theorem 1. Let $(G(-), D)$, G, T and A satisfy the hypotheses of Theorem 1. By Lemma 4.1, the G-map $v \mapsto G_v$ is a bijection between the vertex set VT and the set of vertex stabilizers in G. In particular, elements of the center of G must act trivially on VT. Thus, T_0 is an $InnG$-tree (with every inner automorphism i_g having the same action as $g \in G$) and the bijection induces an action of A on VT which extends the action of $InnG$. Of course, the EGI hypothesis is equivalent to the assumption that G (and hence, $InnG$) acts locally edge-transitively on T.

Thus, Lemma 3.1 implies that A induces a group of automorphisms of the undirected tree T_0 and so, by the Bass-Serre theory, A also admits a presentation as the fundamental group of a certain graph of groups $(A(-), D^*)$. The graph D^* in this presentation may be taken to be the quotient T^*/A, where T^* is the directed A-tree obtained from T by subdividing each inverted edge of T and orienting both newly created edges as outgoing from the new vertex.

Indeed, A induces a group of automorphisms of the undirected quotient graph $D_0 = T_0/G$ and, because A contains the group $InnG$ of inner automorphisms of G, an edge of T_0 is inverted by an element of A if and only if the projection of this edge in D_0 is inverted by some element of A. For if e is an edge of T_0 incident with vertices u and v and if $\alpha \in A$ inverts the projection e^G of e in D_0, then the ordered pair (u^α, v^α) is G-conjugate to the pair (v, u). Thus, αi_g inverts e for some $g \in G$. It follows that D_0^* can be described in terms of D_0 (that is, without reference to T) as the graph obtained by subdividing the edges of D_0 that are inverted by elements of A (to get an inversion-free A-graph) and then forming the quotient graph modulo A.

Because $(G(-), D)$ is proper, the G-stabilizer of each edge of T_0 is properly contained in the G-stabilizer of both incident vertices unless those vertices are G-conjugate. Since $InnG \subseteq A$, a similar statement holds for the action of A on T_0. It remains true for the action of A on T^* and so $(A(-), D^*)$ is proper. That $(A(-), D^*)$ also satisfies the EGI hypothesis is immediate from Lemma 3.2.

Statement (ii) of Theorem 1 is implicit in the proof above and the edge number inequality of (iii) follows because the "new" edges produced by subdivision of an A-inverted edge are A-conjugate. It only remains to verify the second statement in (iii).

Suppose H is a group acting as isometries of a finite directed tree X (with no edge inversions). Arguing by induction on $|VX|$, we claim that the quotient graph X/H is also a tree. For because X is finite, H fixes some vertex v of X and the subtree $star(v)$ spanned by $\{u \in VX : d(u, v) = 1\}$ is then H-invariant. If X_0 is the H-tree obtained from X by contracting this subtree, then X_0/H is a tree by the inductive hypothesis and thus, X/H is also a tree. This observation applied in the case $X = D$ and $H = A$ implies that D^* is a tree if D is a finite tree, thus completing the proof of Theorem 1.

Theorem 1 applied to the case of a proper HNN extension yields an analog of the Karrass-Pietrowski-Solitar theorem. (See also Theorem 3.3 of [5].)

Theorem 2 *Let $G = H *_U t$ be an HNN extension with $H \neq U$ and assume that the subgroup U is conjugate maximal in H. Let A be the subgroup of $AutG$ consisting of all automorphisms which map H to a conjugate of itself. If $B = N_{AutG}(H) \cap N_{AutG}(H^{t^{-1}})$, then either*

(i) $|N_{AutG}(H) : B| = |H : U|$ *and A is an HNN extension $N_{AutG}(H) *_B i_t$ (where i_t is the inner automorphism of G induced by t) or*

(ii) $|N_{AutG}(H) : B| = 2|H : U|$ *and A is an amalgamated free product $N_{AutG}(H) *_B D$ where $D = B\langle\delta\rangle$ and δ interchanges H and $H^{t^{-1}}$ (whence $|D : B| = 2$).*

The latter case occurs if and only if H admits an automorphism β with $U^\beta = U^t$ such that the map $tt^\beta : U \to H$ (where $t^\beta = \beta^{-1}t\beta$) is the restriction to U of an inner automorphism of H.

Proof. Assume that G is an HNN extension $H *_U t$ with $H \neq U$ and U conjugate maximal. Here, D is a loop and A is the group of automorphisms of G which map H to a conjugate of itself. Again the hypotheses of Theorem 1 are satisfied. The action of G on T is both vertex and edge-transitive and so, since $A \supseteq InnG$, this is certainly true of the action of A. Therefore, if the action of A on T is free of inversions and if $p \in VT$ with $H = G_p$ then $|A_p : B| = |H : U|$ and A is an HNN-extension $A_p *_B i_t$, where i_t is the inner automorphism induced by t and $B = A_p \cap (A_p)^{i_t^{-1}}$. On the other hand, if there is an inversion then the action of A on the barycentric subdivision T^* of T yields the decomposition $A = A_p *_B A_r$, where r is the mid-point of the edge $[p, p^{t^{-1}}]$. Note that each vertex p of T has $|H : U|$ outgoing edges (all conjugate in G_p) and the same number of incoming edges (again all conjugate in G_p). However, in T^* all the (new) edges incident at p have the same orientation relative to p and, because A inverts an edge of T, they are all conjugate in A_p. Thus, in this case $|A_p : B_p| = 2|H : U|$.

Finally, we claim that the action of A contains an inversion if and only if H admits an automorphism β such that $U^\beta = U^t$ and, for some $a \in H$, $u^{t\beta^{-1}t\beta} = u^a$ for all $u \in U$. For with such an automorphism we can define $\alpha \in A$ by setting $h^\alpha = (h^{\beta^{-1}})^t$ for all $h \in H$ and $t^\alpha = t^{-1}a^{\beta^{-1}}$. Since α interchanges H and H^t, it inverts the edge $[p, p^t]$ in T. Conversely, if α is an automorphism of G which inverts $[p, p^t]$, then it interchanges H and H^t, and so if $\gamma = i_t\alpha^{-1}$, then $\gamma|_H \in AutH$ and $H^{t^{-1}\gamma} = H^{\alpha^{-1}} = H^t$. If $\beta = \gamma|_H$ then, because $U = H \cap H^{t^{-1}}$, it follows that $U^\beta = U^\gamma = U^t$. If $a = tt^{\alpha^{-1}}$, then

$$H^a = H^{t\alpha t\alpha^{-1}} = H$$

and so $a \in N_G(H) = H$. Moreover, for any $u \in U$,

$$u^{tt^\beta} = u^{t\beta^{-1}t\beta} = u^{t\gamma^{-1}t\gamma} = u^{t\alpha t\alpha^{-1}} = u^a$$

and so the proof of Theorem 2 is complete.

In general, while Theorem 1 specifically identifies the vertex groups of $(A(-), D^*)$ as subgroups of A, in the absence of further hypotheses about $(G(-), D)$ it seems difficult to obtain structural information about these groups. However, additional assumptions about the $(G(-), D)$ may make such information accessible and yield useful conclusions about the structure of A. The remarks in Section 5 provide one illustration of this. In a different vein, if the vertex groups of $(G(-), D)$ in Theorem 1 are all finite then Corollary 5.12 of [3] implies that those of $(A(-), D^*)$ are also finite and so, if D is a finite graph (or more generally, if the orders of the vertex groups are bounded) then A (that is, $AutG$) is virtually free. (See Theorem IV.1.6 of [2].) D. Varsos has kindly brought to our attention his recent work (with E. Raptis and O. Talelli) on automorphism groups of graph products and, in particular, to [5] which concerns automorphism groups of amalgamated free products and HNN extensions of polycyclic-by-finite groups. Using results from [6] and [3], the proof of the main result of [5] is easily adapted to yield the following generalization:

Theorem 3 *Let G and D be as in Theorem 1 and suppose that D is finite. Assume that for any vertex $\mathbf{v}$ of D, the vertex group $G(\mathbf{v})$ is polycyclic-by-finite and that each $\mathbf{v}$-incident edge group has finite index in $G(\mathbf{v})$. If G is $\mathbf{Z}$-linear, then $AutG$ is abelian-by-$\mathbf{Z}$-linear. If G is $\mathbf{Z}$-linear and in addition, in each vertex group the extraction of roots is unique (e.g. if each vertex group is torsion-free nilpotent), then $AutG$ is $\mathbf{Z}$-linear.*

Proof. Except for the use of Corollary 5.12 of [3], the proof is identical to that of Corollary 3.5 of [5] which it generalizes. By Theorem 1.3 of [6], the $\mathbf{Z}$-linearity of G implies that the kernel K of the action of G on the standard tree has finite index in each edge group. G/K is, therefore, the fundamental group of a finite graph of finite groups and in particular, is finitely generated and virtually free. But this graph of finite groups satisfies the hypotheses of Corollary 5.12 of [3] and so $Out(G/K)$ is finite and $Aut(G/K)$ is finitely generated and virtually free. In particular, $Aut(G/K)$ is $\mathbf{Z}$-linear. $AutK$ is also $\mathbf{Z}$-linear because K is polycyclic-by-finite. But K is characteristic in G (by Corollary 2.4 of [5]) and so there is a homomorphism from $AutG$ to the $\mathbf{Z}$-linear group $Aut(G/K) \times AutK$. Since the kernel is abelian (see, for example, Proposition 2.5 of [5]), the first conclusion of the theorem holds. As in the proof of Corollary 3.6 of [5], if the extraction of roots in the vertex groups is unique, then this kernel is trivial and the second conclusion follows.

5 On a class of HNN extensions

Let U be a proper subgroup of H and let $G = H *_U t$ where we assume that $u^t = u$ for all $u \in U$. As before, let A be the group of automorphisms of G

which map H to a conjugate of itself. Here, using Theorem 2 and a lemma from Section 2, we record a complete description of the structure of A (in terms of H and U). (It seems very unlikely that these observations are new but we are not aware of a convenient reference, even for the case $U = \{1\}$.)

Since t centralizes U, the final statement of Theorem 2 (with $\beta = id_H$) implies that A is an amalgamated free product $N_{AutG}(H) *_B D$ where $B = N_{AutG}(H) \cap N_{AutG}(H^{t^{-1}})$ and $D = B\langle\delta\rangle$ for some δ. In fact, for δ we may take the automorphism of order 2 defined by $h \mapsto h^{t^{-1}}$ $(\forall h \in H)$ and $t \mapsto t^{-1}$. If $a \in G$, let i_a be the inner automorphism of G induced by (conjugation by) a and if $a \in N_G(H)$, let $j_a = i_a|_H \in AutH$. If $X \leq G$, let $Inn_X(G) = \{i_x : x \in X\}$. Also, let θ be the automorphism of G which fixes elements of H and inverts t.

We first analyze the factor $N_{AutG}(H)$.

If α is any element of $N_{AutG}(H)$ then by Lemma 2.6, $t^\alpha = a^{-1}t^\epsilon b$ for some $a, b \in H$ and $\epsilon \in \{-1, 1\}$ and hence, for some $\beta \in \langle\theta\rangle$, $t^{\alpha\beta} = a^{-1}tb$. Then

$$u^\alpha = u^{\alpha\beta} = (u^t)^{\alpha\beta} = (u^{\alpha\beta})^{t^{\alpha\beta}} = (u^\alpha)^{a^{-1}tb}$$

and so $(u^\alpha)^{a^{-1}} \in H \cap H^{t^{-1}} = U$ for any $u \in U$. Thus, $(U^\alpha)^{a^{-1}} \subseteq U$. A similar argument using $(\alpha\beta)^{-1}$ instead of $\alpha\beta$ yields that $(U^a)^{\alpha^{-1}} \subseteq U$ and so $\alpha i_a^{-1} \in N_{AutG}(U)$. Moreover, since t centralizes U, $(u^\alpha)^{a^{-1}tb} = (u^\alpha)^{a^{-1}b}$ and so $a^{-1}b \in C_H(U^\alpha) = C_H(U^a)$, whence $c = ab^{-1} \in C_H(U)$.

If $\gamma = (\alpha i_a^{-1})|_H \in N_{AutH}(U)$, let $\phi_{(\gamma,c)}$ be the automorphism of G which maps h to h^γ for all $h \in H$ and sends t to tc^{-1}. Then $\alpha = \phi_{(\gamma,c)} i_a \beta = \phi_{(\gamma,c)} \beta i_a$. This motivates the following construction:

Let E_0 be the semidirect product $N_{AutH}(U)[C_H(U)]$ (using the obvious action). Elements (γ, c) of E_0 are identified with automorphisms $\phi_{(\gamma,c)}$ as above and this defines a homomorphism from E_0 to $N_{AutG}(H) \cap N_{AutG}(U)$. By virtue of this homomorphism, E_0 acts on $Inn_H(G)$. (In fact, under this action, $C_H(U)$ centralizes $Inn_H(G)$ while $N_{AutH}(U)$ acts in the obvious way.) Thus, we may form a second semidirect product $E_1 = E_0[Inn_H(G)]$. A simple computation yields that

$$\theta^{-1}(\phi_{(\gamma,c)} i_a)\theta = \phi_{(\gamma j_c, c^{-1})} i_{c^{-1}a}$$

and we may lift this to an action of $\langle\theta\rangle$ on E_1 (which is trivial on $Inn_H(G)$) and construct a corresponding third semidirect product $E = [E_1]\langle\theta\rangle$.

As observed above, the homomorphism $E \to N_{AutG}(H)$ defined by

$$(\gamma, c, i_a, \beta) \mapsto \phi_{(\gamma,c)} i_a \beta$$

is surjective. If (γ, c, i_a, β) is in the kernel, then $a^{-1}t^\beta c^{-1}a = t$. If $\beta = \theta$, then $(ta)^2 = c^{-1}a^2 \in H$ which is false. Thus, $\beta = id_G$, whence

$$a^t = c^{-1}a \in H \cap H^t = U^t$$

and so $a \in U$, $\gamma = j_a^{-1}$ and $c = 1$. We conclude that

$$N_{AutG}(U) = E^\phi \cong E/K$$

where $K = \{(j_a^{-1}, 1, i_a, id_G) : a \in U\}$.

Next we consider the amalgamated subgroup $B = N_A(H) \cap N_A(H^{t^{-1}})$. Certainly $B \subseteq N_{AutG}(H \cap H^{t^{-1}}) = N_{AutG}(U)$ and so if $\alpha \in B$ then

$$t^{\alpha} \in C_G(U^{\alpha}) = C_G(U)$$

. But also, $H^{t^{-1}} = (H^{t^{-1}})^{\alpha} = H^{t^{-\alpha}}$ and so

$$t^{-1}t^{\alpha} \in N_G(H) \cap C_G(U) = C_H(U)$$

. Therefore, if $\gamma = \alpha|_H \in N_{AutH}(U)$ and $t^{-1}t^{\alpha} = c^{-1}$, then

$$\alpha = \phi_{(\gamma,c)} \in E_0$$

. It follows that $B = (E_0)^{\phi} \cong N_{Aut(H)}(U)[C_H(U)]$.

It is easily checked that δ centralizes each $\phi_{(\gamma,1)}$ and if $c \in C_H(U)$, then

$$\delta^{-1}\phi_{(id_H,c)}\delta = \phi_{(j_c,c^{-1})}.$$

This determines the structure of D as a semidirect product $[B]\langle\theta\rangle$.

Thus, the structure of the factors of A and of the amalgamated subgroup are completely specified in terms of H and U.

Because $H \cap Z(G) \subseteq C_H(t) = U$, $H \cap Z(G) = U \cap Z(G)$ and so

$$|Inn_H(G) : Inn_U(G)| = |H : U|$$

. Since $K \cong Inn_U(G)$, it follows that if H is finite, then A (which is all of $AutG$ in this case) is an amalgamated free product of finite groups of order $2|N_{AutH}(U)||C_H(U)||H : U|$ and $2|N_{AutH}(U)||C_H(U)|$ with the amalgamated subgroup of order $|N_{AutH}(U)||C_H(U)|$.

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